PERIODIC HOMOGENIZATION WITH AN INTERFACE: THE ONE-DIMENSIONAL CASE

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Abstract. We consider a one-dimensional diffusion process with coefficients that are periodic outside of a finite 'interface region'. The question investigated in this article is the limiting long time / large scale behaviour of such a process under diffusive rescaling. Our main result is that it converges weakly to a rescaled version of skew Brownian motion, with parameters that can be given explicitly in terms of the coefficients of the original diffusion.

Our method of proof relies on the framework provided by Freidlin and Wentzell [FW93] for diffusion processes on a graph in order to identify the generator of the limiting process. The graph in question consists of one vertex representing the interface region and two infinite segments corresponding to the regions on either side.

1. Introduction

Consider a diffusion process in $\mathbb{R}^d$ of the type

$$dX(t) = b(X)\,dt + dB(t)\;, \quad X(0) \in \mathbb{R}^d\;, \tag{1.1}$$

where B is a d-dimensional Wiener process and $b\colon \mathbb{R}^d \to \mathbb{R}^d$ is periodic and smooth, and define the diffusively rescaled process $X^\varepsilon(t) = \varepsilon X(t/\varepsilon^2)$. If b is periodic and satisfies a natural centering condition, then it is well-known that X^ε converges in law as $\varepsilon \to 0$ to a Wiener process with a 'diffusion tensor' that can be expressed in terms of the solution to a suitable Poisson equation, see for example the monographs [BLP78, PS08].

Similar types of homogenization results still hold true if b is not exactly periodic, but of the form $b(X) = \tilde b(X, \varepsilon X)$, for some smooth function $\tilde b$ that is periodic in its first argument. In other words, b consists of a slowly varying component, modulated by fast oscillations. In this case, the limiting process is not a Brownian motion anymore, but can be an arbitrary diffusion, whose coefficients can again be obtained by a suitable averaging procedure [BMP05]. The aim of this article is to consider a somewhat different situation where there is an abrupt change from one type of periodic behaviour to another, separated by an interface of size order one in the original 'microscopic' scale.

To the best of our knowledge, this situation has not been considered before, although a similar problem was studied in [ACP03]. In order to keep calculations simple, we restrict ourselves to the one-dimensional situation, and we plan to address the multidimensional case in a future publication. This considerably simplifies the analysis due to the following two facts:

- Any one-dimensional diffusion is reversible, so that its invariant measure can be given explicitly.

- The 'interface' is a zero-dimensional object, so that it cannot exhibit any internal structure in the limit.

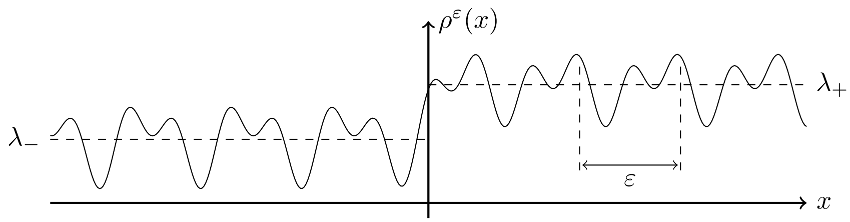
Before we give a more detailed description of our results, let us try to 'guess' what any limiting process X^0 should look like, if it exists. Away from the interface, we can apply the existing results on periodic homogenization, as in [BLP78, PS08]. We can therefore compute diffusion coefficients $C_\pm$ such that X^0 is expected to behave like $C_+W(t)$ whenever $X^0 > 0$ and like $C_-W(t)$ whenever $X^0 < 0$, for some Wiener process W. One possible way of constructing a Markov process with this property is to take $X^0(t) = G(W(t))$, where $G\colon \mathbb{R} \to \mathbb{R}$ is given by

$$G(x) = \begin{cases} C_+x & \text{if } x \geq 0, \\ C_-x & \text{otherwise.} \end{cases}$$

In turns out that processes of this form do not describe all the possible limiting processes that one can get in the presence of an interface. The reason why this is so can be seen by comparing the invariant measure of $g(W(t))$ to the invariant measure of X^ε. Since the invariant measure for W is Lebesgue measure (or multiples thereof), the invariant measure for $G(W(t))$ is given by

$$\mu(dx) = \begin{cases} \lambda_+\, dx & \text{if } x > 0, \\ \lambda_-\, dx & \text{otherwise.} \end{cases} \quad , \quad \text{with} \quad \frac{\lambda_+}{\lambda_-} = \frac{C_-}{C_+}\ . \tag{1.2}$$

On the other hand, if we denote the invariant measure for X^ε by $\mu^\varepsilon = \rho^\varepsilon(x)\,dx$, then ρ^ε will typically look as follows:



It follows that one does indeed have $\mu^\varepsilon \to \mu^0$ as $\varepsilon \to 0$, where μ^0 is of the type (1.2), but the ratio λ_+/λ_- depends on the behaviour of b, not only away from the interface, but also at the interface. This can be understood as the process X^0 picking up an additional drift, proportional to the local time spent at 0, that skews the proportion of time spent on either side of the interface. A Markov process with these properties can be constructed by applying the function G to a skew-Brownian motion (see for example [Lej06]) with parameter p for a suitable value of p.

An intuitive way of constructing this process goes as follows. First, draw the zeroes of a standard Wiener process on the real line. These form a Cantor set that partitions the line into countably many disjoint open intervals. Order them by decreasing length and denote by I_n the length of the nth interval. For each $n \geq 0$, toss an independent biased coin and draw an independent Brownian excursion. If the coin comes up heads (with probability p), fill the interval with the Brownian excursion, scaled horizontally by $1/I_n$ and vertically by $C_+/\sqrt{I_n}$. Otherwise (with probability $1-p$), fill the interval with the Brownian excursion, scaled horizontally by $1/I_n$ and vertically by $-C_-/\sqrt{I_n}$. One can check that the invariant measure for this process is given by (1.2), but with

$$\frac{\lambda_+}{\lambda_-} = \frac{p\,C_-}{(1-p)\,C_+}\ . \tag{1.3}$$

We denote the corresponding process by $B_{C_\pm,p}(t)$.

This should almost be sufficient to guess the main result of this article. To fix notations, we consider the process $X(t)$ as in (1.1) and its rescaled version X^ε, and we assume that the drift function b is smooth and periodic away from an 'interface' region $[-\eta,\eta]$. More precisely, we assume that there exist smooth periodic functions $b_i\colon \mathbb{R}\to\mathbb{R}$, $i\in\{+,-\}$, such that $b_i(x+1)=b_i(x)$ and such that $b(x)=b_+(x-\eta)$ for $x>\eta$ and $b(x)=b_-(x+\eta)$ for $x<-\eta$. Additionally, we assume that the functions b_i satisfy the centering condition

$$\int_0^1 b_i(x)\,dx = 0\;.$$

We also set $V(x)=\int_0^x b(x)\,dx$ for $x\in\mathbb{R}$, so that $\exp(2V(x))\,dx$ is invariant for X, and similarly for V_i. Denote by C_i the effective diffusion coefficients for the periodic homogenization problems corresponding to b_i (see equation (3.5) below or [PS08] for a more explicit expression). Define furthermore $\lambda_\pm$ by

$$\lambda_+ = \int_\eta^{\eta+1}\exp(2V(x))\,dx\;,\qquad \lambda_- = \int_{-\eta-1}^{-\eta}\exp(2V(x))\,dx\;, \tag{1.4}$$

and let $p\in(0,1)$ be the unique solution to (1.3). With all these notations at hand, we have:

Theorem 1.1. *For any $t>0$, the law of X^ε converges weakly to the law of $B_{C_\pm,p}$ in the space $\mathcal{C}([0,t],\mathbb{R})$.*

Remark 1.2. In order to keep notations simple, we have assumed that the diffusion coefficient of X is constant and equal to 1. The case of a non-constant, but smooth and uniformly elliptic diffusion coefficient can be treated in exactly the same way, noting that it reduces to the case treated here after a time change that can easily be controlled.

The proof of the weak convergence of the probability distributions on $\mathcal{C}[0,\infty)$ associated to X_x^ε presented in this article will rely heavily on the 1993 paper by Freidlin and Wentzell [FW93], in which the authors consider a 'fast' Hamiltonian system perturbed by a 'slow' diffusion. Theorems 2.1 and 4.1 from [FW93] provide a general framework for proving first the tightness and then the convergence of a family of probability distributions on $\mathcal{C}[0,\infty)$.

We will start by showing tightness of our family of processes in Section 2. Once tightness is established, we show in Section 3 that every limiting process X^0 solves the martingale problem associated to a certain generator. Finally, we show in Section 4 that this martingale problem has a unique solution which is precisely the rescaled skew-Brownian motion, thus concluding the proof.

2. Proving tightness

The main result of this section is the following:

Theorem 2.1. *The family of probability measures on $\mathcal{C}_0^\infty(\mathbb{R})$ given by the laws of X_x^ε for $\varepsilon\in(0,1]$ is tight.*

As usual in the theory of homogenization, we show this by first introducing a compensator $g\colon\mathbb{R}\to\mathbb{R}$ that 'kills' the strong drift of the rescaled process and looking at the process

$$Y^\varepsilon(t) = X^\varepsilon(t) + \varepsilon g\Big(\frac{X^\varepsilon(t)}{\varepsilon}\Big) . \tag{2.1}$$

Since we will choose g to be a bounded function, the tightness of the laws of X^ε is equivalent to that of the Y^ε (see Proposition 2.5 below), so that it remains to show tightness of Y^ε. For this, we will adapt a proof from [SV79], but the argument will have to be modified in order to take into account the behaviour of the process at the interface.

In order to construct g, let $\mathcal{L}_i$ denote the generator of the diffusion with drift b_i, that is $\mathcal{L}_i = \frac{1}{2}\partial_x^2 + b_i(x)\partial_x$, and denote by $\mu_i(dx) = Z^{-1}\exp(2V_i(x))\,dx$ the corresponding invariant probability measure on $[0,1]$. We then denote by g_i the unique smooth function solving

$$\mathcal{L}_i g_i = -b_i \;, \qquad \int_0^1 g_i(x)\,\mu_i(dx) = 0 \;. \tag{2.2}$$

Such a function exists by the Fredholm alternative. We now choose any smooth function $g\colon \mathbb{R}\to\mathbb{R}$ such that $g(x) = g_-(x+\eta)$ for $x\in(-\infty,\eta)$ and $g(x) = g_+(x-\eta)$ for $x\in(\eta,\infty)$, with a smooth joining region in between (the function g will be chosen later to satisfy an additional condition in the joining region, but this is not necessary for the tightness proof).

With these notations, the main result of this section is given by:

Theorem 2.2. *For fixed x, the laws of Y_x^ε for $\varepsilon\in(0,1]$ form a tight family of measures on $\mathcal{C}([0,\infty),\mathbb{R})$.*

The proof of this result will be given below, as a consequence of the following more general statement:

Theorem 2.3. *Consider a family of probability measures $\mathscr{P}$ on $\Omega = \mathcal{C}([0,\infty),\mathbb{R}^d)$ such that*

$$\lim_{l\nearrow\infty}\sup_{\mathbb{P}\in\mathscr{P}}\mathbb{P}\big(\,|x(0,\omega)|\ge l\,\big) = 0 \;,$$

where, for $\omega\in\Omega$, we denoted by $x(t,\omega)$ the canonical process on $\mathcal{C}([0,\infty),\mathbb{R}^d)$. Denote furthermore by $\mathscr{F}_t$ the natural filtration of Ω, that is $\mathscr{F}_t$ is the σ-algebra generated by $x(s)$ for $s\le t$.

For every $\rho>0$, fix a positive number A_ρ and a function $f_\rho\in\mathcal{C}_0^\infty(\mathbb{R}^d,\mathbb{R})$ such that $0\le f_\rho\le 1$, $f_\rho(0)=1$, and $f_\rho = 0$ for $|x|\ge c\rho$ for some fixed $c\le 1$. Denote furthermore by f_ρ^a the translated function $f_\rho^a = f_\rho(x-a)$, and denote by $S(\rho)$ the set of values a such that $t\mapsto f_\rho^a(x(t)) + A_\rho t$ is an $\mathscr{F}_t$-submartingale under any probability measure $\mathbb{P}\in\mathscr{P}$.

With these objects at hand if, for any $\rho>0$, one can find a series of stopping times τ_n^ρ, $\tau_0^\rho = 0$, with $\lim_{n\to\infty}\tau_n^\rho = \infty$ ($\mathbb{P}$-almost surely for any $\mathbb{P}\in\mathscr{P}$) such that

$$|x(\tau_n^\rho) - x(\tau_{n+1}^\rho\wedge(t+\tau_n^\rho))| \le \rho, \quad c\rho \le |x(\tau_n^\rho) - x(\tau_{n+1}^\rho)| \tag{2.3}$$

(again $\mathbb{P}$-almost surely for any $\mathbb{P}\in\mathscr{P}$) for all $t\ge 0$, $n\ge 0$, and such that $x(\tau_n^\rho)\in S(\rho)$, then the family $\mathscr{P}$ is tight.

Proof. The proof is very similar to that of [SV79, Theorem 1.4.6]. The main difference is that we have introduced a sequence $S(\rho)$ of 'good' subsets of $\mathbb{R}^d$ on which we can control the speed at which the process moves. In the original proof, these subsets were taken to be the whole space $\mathbb{R}^d$. It is known (see for example

[SV79, Theorem 1.3.1]), that tightness of $\mathscr{P}$ follows if one can show that, for every $0 < T < \infty$ and every $\rho > 0$, one has

$$\lim_{\delta\searrow 0} \inf_{\mathbb{P}\in\mathscr{P}} \mathbb{P}\Big(\sup_{0\le s< t\le T,\ |t-s|<\delta} |x(t) - x(s)| \le \rho \Big) = 1 \;. \tag{2.4}$$

Let now N be the random quantity given by $N = \min\{n : \tau^\rho_{n+1} > T\}$ and let $\eta(\rho) = \min\{\tau^\rho_n - \tau^\rho_{n-1} : 1 \le n \le N\}$. Let furthermore t_1 and t_2 be any pair of times in $[0,T]$ such that $|t_2 - t_1| \le \eta(\rho)$. Then

$$\sup\{\, |x(t_2) - x(t_1)| : 0 \le t_1 < t_2 \le T \text{ and } |t_2 - t_1| \le \eta(\rho)\} \;\le 4\rho \;.$$

This can be seen by considering the partition of $[0,T]$ into the subintervals $[\tau^\rho_0, \tau^\rho_1)$, $\ldots$, $[\tau^\rho_{N-1}, \tau^\rho_N)$, and $[\tau^\rho_N, T]$. All of these subintervals, except possibly the last one, must have length greater than $\eta(\rho)$. Thus, either both t_1 and t_2 lie in the same subinterval, or they lie in adjacent subintervals. Since over any subinterval the the distance of the path from its left hand end never exceeds ρ, the oscillation of the path over any subinterval must be less than or equal to 2ρ. Hence the oscillation over the union of any two successive subintervals cannot exceed 4ρ, and in particular $|x(t_2) - x(t_1)| \le 4\rho$.

Combining this observation with (2.4) and the fact that ρ was arbitrary in this argument, we see that tightness of $\mathscr{P}$ is implied by

$$\lim_{\delta\searrow 0} \sup_{\mathbb{P}\in\mathscr{P}} \mathbb{P}\big(\{\, \omega : \eta(\rho) \le \delta\}\big) = 0 \;, \tag{2.5}$$

so that our problem is reduced to getting upper bounds on $\eta(\rho)$ that are independent of $\mathbb{P} \in \mathscr{P}$. Let us fix any such $\mathbb{P}$ from now on.

The main step in obtaining such a bound is to realise that the properties of the τ^ρ_n imply that, for any $\delta > 0$, any $\rho > 0$, and any $n \ge 0$, one has the bound

$$\mathbb{P}(\, \tau^\rho_{n+1} - \tau^\rho_n \le \delta | \mathscr{F}_{\tau^\rho_n}) \le \delta A_\rho, \qquad \mathbb{P} \text{ a.s, on } \{\tau^\rho_n < \infty\} \;. \tag{2.6}$$

As in [SV79, Lemma 1.4.4], the bound (2.6) is obtained consequence of the fact that a submartingale can be stopped and started at stopping times without losing the submartingale property (see [SV79, Theorems 1.2.5 and 1.2.10] for a more precise formulation of this statement). In particular, since we assumed that $x(\tau^\rho_n) \in S(\rho)$, the identity

$$\mathbb{E}\Big(f_\rho^{x(\tau^\rho_n)}\big(x\big(\tau^\rho_{n+1} \wedge (\tau^\rho_n + \delta)\big)\big) + \delta A_\rho \,\Big|\, \mathscr{F}_{\tau^\rho_n}\Big) \ge f_\rho^{x(\tau^\rho_n)}\big(x(\tau^\rho_n)\big) = 1 \;, \tag{2.7}$$

holds $\mathbb{P}$-almost surely. On the other hand, since f_ρ is bounded by 1 and one has $f_\rho^{x(\tau^\rho_n)}\big(x(\tau^\rho_{n+1})\big) = 0$ from the properties of f_ρ and the τ_n, one has the bound

$$\begin{aligned}
\mathbb{E}\Big(f_\rho^{x(\tau^\rho_n)}&\big(x\big(\tau^\rho_{n+1} \wedge (\tau^\rho_n + \delta)\big)\big) + \delta A_\rho \,\Big|\, \mathscr{F}_{\tau^\rho_n}\Big) \\
&\le (1 + \delta A_\rho)\mathbb{P}(\tau^\rho_{n+1} > \delta + \tau^\rho_n | \mathscr{F}_{\tau^\rho_n}) + \delta A_\rho \mathbb{P}(\, \tau^\rho_{n+1} \le \delta + \tau^\rho_n | \mathscr{F}_{\tau^\rho_n}) \\
&= \delta A_\rho + \mathbb{P}(\, \tau^\rho_{n+1} - \tau^\rho_n > \delta | \mathscr{F}_{\tau^\rho_n}) \\
&= \delta A_\rho + 1 - \mathbb{P}(\, \tau^\rho_{n+1} - \tau^\rho_n \le \delta | \mathscr{F}_{\tau^\rho_n}) \;.
\end{aligned}$$

Combining this with (2.7), the estimate (2.6) follows.

Let N be again as above and note that from the definition of $\eta(\rho)$ we have the estimate

$$\mathbb{P}\big(\eta(\rho) \le \delta\big) \le \mathbb{P}\big(\min\{\tau_i - \tau_{i-1} : 1 \le i \le k\} \le \delta\big) + \mathbb{P}(N > k)$$

$$\leq \sum_{i=1}^{k} \mathbb{P}(\tau_i - \tau_{i-1} \leq \delta) + \mathbb{P}(N > k) \leq k\delta A_\rho + \mathbb{P}(N > k) ,$$

where we used (2.6) to obtain the last line. Thus the proof of (2.5) will be complete once it is established that

$$\lim_{k\nearrow\infty} \sup_{\mathbb{P}\in\mathscr{P}} \mathbb{P}(N > k) = 0 . \tag{2.8}$$

This will turn out to be a consequence of the following general fact, that can be found in [SV79, Lemma 1.4.5]:

Lemma 2.4. *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and $\{\mathscr{F}_n : n \geq 0\}$ be a non-decreasing sequence of sub σ-algebras of $\mathscr{F}$. Let $\{\tau_n : n \geq 1\}$ be a non-decreasing sequence of random variables on $(\Omega, \mathscr{F})$ taking values in $[0,\infty) \cup \{\infty\}$, and assume that τ_n is $\mathscr{F}_n$ measurable. Define $\tau_0 \equiv 0$ and suppose for some $\lambda < 1$ and all $n \geq 0$:*

$$\mathbb{E}\big[\exp[-(\tau_{n+1} - \tau_n)] \,|\, \mathscr{F}_n\big] \leq \lambda \text{ a.s.}$$

If, for some $T > 0$ one defines for $\omega \in \Omega$,

$$N(\omega) \;=\; \inf\{\, n \geq 0 : \tau_{n+1}(\omega) > T\}$$

then $N < \infty$ a.s. and in fact one has the stronger bound $\mathbb{P}(N \geq k) \;\leq e^T \lambda^k$. □

In order to apply this result, note that for any $t_0 > 0$, one has the $\mathbb{P}$-almost sure inequality

$$\begin{aligned}\mathbb{E}\,[\, e^{-(\tau_{i+1}-\tau_i)} | \mathscr{F}_{\tau_i}] \;&\leq\; \mathbb{P}\,[\,\tau_{i+1} - \tau_i \leq t_0 | \mathscr{F}_{\tau_i}] + e^{-t_0}\mathbb{P}\,[\,\tau_{i+1} - \tau_i > t_0 | \mathscr{F}_{\tau_i}]\\ &\leq\; e^{-t_0} + (1 - e^{-t_0})\mathbb{P}\,[\,\tau_{i+1} - \tau_i \leq t_0 | \mathscr{F}_{\tau_i}]\\ &\leq\; e^{-t_0} + (1 - e^{-t_0}) t_0 A_\rho .\end{aligned}$$

Choosing t_0 sufficiently small so that $t_0 A_\rho < 1$, we can make this strictly smaller than 1. Therefore, Lemma 2.4 yields the existence of $\lambda > 0$ independent of $\mathscr{P}$ such that

$$\mathbb{P}(N \geq k) \;\leq\; e^T \lambda^k ,$$

from which (2.8) follows at once. □

This general statement can now be applied to the situation that is of interest to us, allowing us to give the

Proof of Theorem 2.2. Denoting by $\mathcal{L} = \frac{1}{2}\partial_x^2 + b(x)\partial_x$ the generator of the diffusion X, it follows from Itô's formula that Y^ε satisfies the equation

$$Y_x^\varepsilon(t) = x + \frac{1}{\varepsilon}\int_0^t (b + \mathcal{L}g)\Big(\frac{X^\varepsilon(s)}{\varepsilon}\Big)\, ds + \int_0^t \Big(1 + g'\Big(\frac{X^\varepsilon(s)}{\varepsilon}\Big)\Big)\, dB_s ,$$

where, as a consequence of the construction of g, the modified drift $b + \mathcal{L}g$ vanishes as soon as X is located outside of the interface $[-\eta, \eta]$.

The series of functions and stopping times satisfying the hypotheses of the above theorem are then given as follows. Set $\mathbb{Z}_\rho = \rho\mathbb{Z} + \frac{\rho}{2}$ and define

$$\tau_{n+1}^\rho \;= \inf\{\, t > \tau_n^\rho(\omega) : x(t,\omega) \in \mathbb{Z}_\rho \;, \quad x(t,\omega) \neq x(\tau_n^\rho, \omega)\} .$$

These stopping times clearly satisfy (2.3) with $c = 1$, and we have $\lim_{n\to\infty} \tau_n(\omega) = \infty$ a.s. since Y_x^ε has finite quadratic variation for every $\varepsilon > 0$. Choose furthermore any $\phi \in \mathcal{C}_0^\infty(\mathbb{R})$ having the property that $\phi(x) = 1$ for $|x| \leq \frac{1}{32}$ and $\phi(x) = 0$ for $|x| > \frac{1}{16}$, and set $f_\rho(x) = \phi\big(\frac{x}{\rho}\big)$.

Finally, instead of first fixing the sequence A_ρ, we set $\tilde S(\rho) = (-\infty, -\frac{\rho}{4}] \cup [\frac{\rho}{4}, \infty)$, so that $x(\tau_n^\rho) \in \tilde S(\rho)$ for every ρ. With these definitions at hand, we obtain from Itô's formula

$$\begin{aligned} f_\rho^a(Y^\varepsilon(t)) = f_\rho^a(x) &+ \frac{1}{\varepsilon}\int_0^t (f_\rho^a)'\big(Y^\varepsilon(s)\big)(b+\mathcal{L}g)\big(\varepsilon^{-1}X^\varepsilon(s)\big)\,ds \\ &+ \frac{1}{2}\int_0^t (f_\rho^a)''\big(Y^\varepsilon(s)\big)\Big(1+g'\big(\varepsilon^{-1}X^\varepsilon(s)\big)\Big)^2\,ds \\ &+ \int_0^t (f_\rho^a)'\big(Y^\varepsilon(s)\big)\big(1+g'\big(\varepsilon^{-1}X^\varepsilon(s)\big)\big)\,dB_s\;. \end{aligned}$$

The only term in this expression that is not uniformly bounded with respect to ε is the first term. Recall however that $b + \mathcal{L}g = 0$ outside of $[-\eta, \eta]$ and that f_ρ^a is compactly supported on a ball of radius $\rho/16$ around a. Therefore, provided that $a \in \tilde S(\rho)$, if ε is sufficiently small so that $\varepsilon\eta \le \frac{\rho}{16}$ and $\varepsilon\|g\|_\infty \le \frac{\rho}{16}$, then this first term is identically zero for all times. This shows that one can find a constant A_ρ independent of ε so that the bound

$$f_\rho^a(Y^\varepsilon(t)) \le f_\rho^a(x) + A_\rho t + \int_0^t (f_\rho^a)'\big(Y^\varepsilon(s)\big)\big(1+g'\big(\varepsilon^{-1}X^\varepsilon(s)\big)\big)\,dB_s\;,$$

holds uniformly for all $\varepsilon \in (0,1]$ and all $a \in \tilde S(\rho)$, thus showing that the assumptions of Theorem 2.3 are satisfied for that particular choice of the A_ρ. The initial condition is furthermore kept fixed at one point across the entire family hence the hypothesis on the initial condition is trivially satisfied and the claim follows. □

Finally, we show that Theorem 2.2 does indeed imply Theorem 2.1:

Proposition 2.5. *The tightness of the family of probability measures given by Y^ε and X^ε on $\mathcal{C}([0,\infty),\mathbb{R})$ are equivalent for the same initial conditions.*

Proof. Denote by μ_ε the law of X^ε and by $\tilde\mu_\varepsilon$ the law of Y^ε. Since g is bounded, there exists a constant C such that $d(\mu_\varepsilon, \tilde\mu_\varepsilon) \le C\varepsilon$, where d denotes the Wasserstein-1 metric with respect to the distance function

$$d(x,z) = \sum_{n=1}^\infty \frac{1 \wedge \sup_{0\le t\le n} |x(t) - z(t)|}{2^n}\;.$$

Therefore, since d metrises the topology of weak convergence [Vil03], $\mu^{\varepsilon_n} \to \mu$ weakly for some probability measure μ if and only if $\tilde\mu^{\varepsilon_n} \to \mu$ weakly. Since tightness of a family $\mathscr{P}$ of probability measures over a Polish space is equivalent to the fact that every sequence in $\mathscr{P}$ has at least one (weak) accumulation point, the claim follows. □

3. Convergence of the laws

The main ingredient in our proof of convergence will be [FW93, Theorem 4.1], which is used in conjunction with the previous tightness result to identify the weak limit points of the family of probability distributions as the solutions to a martingale problem. The aim of this section is to explain how to fit our problem into the framework of [FW93] and to verify the assumptions of their main convergence theorem.

Before we proceed, let us recall what is understood by the "martingale problem" corresponding to some operator A (see for example [EK86]), and let us try to guess what the generator A for the limiting process is expected to be. Let X be a Polish (i.e. complete separable metric) space; $\mathcal{C}[0,\infty)$, the space of all continuous functions on $[0,\infty)$ with values in X. For any subset $I \subset [0,\infty)$, denote by $\mathscr{F}_I$ the σ-algebra of subsets of $\mathcal{C}[0,\infty)$ generated by the sets $\{x \in \mathcal{C}[0,\infty) : x(s) \in B\}$, where $s \in I$ and $B \subset X$ is an arbitrary Borel set. We also denote by $\mathcal{C}(X)$ the space of all continuous real-valued functions on X.

Let A be a linear operator on $\mathcal{C}(X)$, defined on a subspace $\mathscr{D}(A) \subseteq \mathcal{C}(X)$. We will say that a probability measure $\mathbb{P}$, on $\big(\mathcal{C}[0,\infty), \mathscr{F}_{[0,\infty)}\big)$, is a solution to the martingale problem corresponding to A, starting from a point $x_0 \in X$, if

$$\mathbb{P}\{x : x(0) = x_0\} = 1 \tag{3.1}$$

and, for any $f \in \mathscr{D}(A)$, the random function defined on the probability space $\big(\mathcal{C}[0,\infty), \mathscr{F}_{[0,\infty)}, \mathbb{P}\big)$ by

$$f\big(x(t)\big) - \int_0^t \big(Af\big)\big(x(s)\big)\,ds,\ t \in [0,\infty)\ , \tag{3.2}$$

is a martingale with respect to the filtration $\big\{\mathscr{F}_{[0,t]}\big\}_{t>0}$.

What do we expect the operator A to be given by in our case? On either side of the interface, we argued in the introduction that the limiting process should be given by Brownian motion, scaled by factors $C_\pm$ respectively. Therefore, one would expect A to be given by

$$\big(Af\big)(x) = \begin{cases} \frac{1}{2}C_-^2\,\partial_x^2 f(x) & \text{if } x < 0, \\ \frac{1}{2}C_+^2\,\partial_x^2 f(x) & \text{otherwise,} \end{cases} \tag{3.3}$$

and the domain $\mathscr{D}(A)$ to contain functions that are $\mathcal{C}^2$ away from the origin. This however does not take into account for the "skewing", which should be encoded in the behaviour of functions in $\mathscr{D}(A)$ at the origin.

Since the limiting process spends zero time at the origin (the invariant measure is continuous with respect to Lebesgue measure), it was shown in [Lej06] that the possible behaviours at the origin are given by matching conditions for the first derivatives of functions belonging to $\mathscr{D}(A)$. We know from the introduction that the invariant measure of the limiting process is proportional to Lebesgue measure on either side of the origin, with proportionality constants $\lambda_\pm$. We should therefore have the identity

$$\lambda_- \int_{-\infty}^0 Af(x)\,dx + \lambda_+ \int_0^\infty Af(x)\,dx = 0\ ,$$

for every function $f \in \mathscr{D}(A)$. Using (3.3), we thus obtain

$$\lambda_- C_-^2 \int_{-\infty}^0 f''(x)\,dx + \lambda_+ C_+^2 \int_0^\infty f''(x)\,dx = 0\ .$$

Integrating by parts, this yields (for say compactly supported test functions f) the condition

$$\lambda_- C_-^2 f'(0^-) = \lambda_+ C_+^2 f'(0^+)\ . \tag{3.4}$$

This is exactly the general form of a generator produced by Theorem 4.1 in Freidlin and Wentzell [FW93], a differential operator on the regions away from some distinguished points termed nodes, combined with a restriction on the ratios of the limits of the derivatives at this point.

The main theorem of this section that is also very closely related to the main theorem of the article is as follows:

Theorem 3.1. *Let $C_\pm$ be given by*

$$C_\pm^2 = \int_0^1 (1 + g_\pm(x))^2 \, \mu_\pm(dx) \;, \tag{3.5}$$

where $g_\pm$ and $\mu_\pm$ are as in (2.2). Let A be given by (3.3) and let $\mathscr{D}(A)$ be the set of continuous functions $f\colon \mathbb{R} \to \mathbb{R}$, vanishing at infinity, that are $\mathcal{C}^2$ away from 0 and that satisfy the condition (3.4) at the origin.

Then, every limit point of the family of processes Y_x^ε is solution to the martingale problem corresponding to A.

As already mentioned, our main ingredient is [FW93, Theorem 4.1] applied to the sequence of processes Y^ε as defined in (2.1). For completeness, we give a simplified statement of this result adapted to the situation at hand:

Theorem 3.2 (Freidlin & Wentzell)**.** *Let $\mathcal{L}_i$, $i = \pm$, be elliptic second order differential operators with smooth coefficients on I_i, $I_+ = [0, \infty)$, $I_- = (-\infty, 0]$, and let Y^ε be a family of real-valued processes satisfying the strong Markov property. For some fixed $\tilde\eta > 0$, let τ^ε be the first hitting time of the set $(-\varepsilon\tilde\eta, \varepsilon\tilde\eta)$ by Y^ε.*

Assume that there exists a function $k\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{\varepsilon\to 0} k(\varepsilon) = 0$ such that, for any function $f \in \mathcal{C}_0^\infty(I_i)$ and for any $\lambda > 0$, one has the bound[1]

$$\mathbb{E}_y \Big[e^{-\lambda\tau^\varepsilon} f\big(Y^\varepsilon(\tau^\varepsilon)\big) - f\big(y\big) + \int_0^{\tau^\varepsilon} e^{-\lambda t}\Big(\lambda f\big(Y^\varepsilon(t)\big) - \mathcal{L}_i f\big(Y^\varepsilon(t)\big)\Big)\, dt \Big] = \mathcal{O}\big(k(\varepsilon)\big) \tag{3.6}$$

as $\varepsilon \to 0$, uniformly with respect to $y \in I_i$. Assume furthermore that there exists a function $\delta\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0$ and $\lim_{\varepsilon\to 0} \delta(\varepsilon)/k(\varepsilon) \to \infty$ such that, for any $\lambda > 0$,

$$\mathbb{E}_y \Big[\int_0^\infty e^{-\lambda t} 1_{(-\delta\,,\,\delta)}\big(Y^\varepsilon(t)\big)\, dt \Big] \to 0 \tag{3.7}$$

as $\varepsilon \to 0$, uniformly over all $y \in R$. Finally, writing σ^δ for the first hitting time of the set $(-\infty, -\delta) \cup (\delta, \infty)$ by Y^ε, assume that there exist $p_i \geq 0$ with $p_- + p_+ = 1$ such that

$$\mathbb{P}_y\big[Y^\varepsilon(\sigma^\delta) \in I_i\big] \to p_i \;, \quad i \in \{+, -\} \;, \tag{3.8}$$

uniformly for $y \in (-\varepsilon\tilde\eta, \varepsilon\tilde\eta)$.

Let now A be the operator defined by $Af(x) = L_i f(x)$ for $x \in I_i$ with domain $\mathscr{D}(A)$ consisting of functions f such that $f|_{I_i} \in \mathcal{C}_0^\infty(I_i)$ and such that the 'matching condition' $p_+ f'(0^+) = p_- f'(0^-)$ holds. Then for any fixed $t_0 \geq 0$, any $\lambda > 0$, and any $f \in \mathscr{D}(A)$, the bound

$$\operatorname{ess\,sup} \Big| \int_{t_0}^\infty e^{-\lambda t} \mathbb{E}_y \Big[\lambda f\big(Y^\varepsilon(t)\big) - A f\big(Y^\varepsilon(t)\big) \Big| \mathscr{F}_{[0,\,t_0]} \Big]\, dt - e^{-\lambda t_0} f\big(Y^\varepsilon(t_0)\big) \Big| \to 0 \tag{3.9}$$

holds as $\varepsilon \to 0$, uniformly for all $y \in \mathbb{R}$.

[1] We denote by $\mathcal{C}_0^\infty$ the space of smooth functions who vanish at infinity, together with all of their derivatives.

Remark 3.3. The version of Theorem 3.2 stated in [FW93] does actually treat more general diffusions on graphs, but assumes that the edges of the graph are finite. This is not really a restriction, since I_+ is in bijection with $[0,1)$ (and similarly for I_-) and we can simply add non-reachable vertices at ± 1 to turn our process into a process on a finite graph.

Remark 3.4. As can be seen by combining (1.3) and (3.4), the probabilities $p_\pm$ appearing in the statement of Theorem 3.2 are not quite the same in general as the probabilities $\tilde p_\pm = \{p, 1-p\}$ appearing in the construction of skew Brownian motion in the introduction. The relation between them is given by $\frac{pC_+}{(1-p)C_-} = \frac{p_+}{p_-}$. The reason is that $p_\pm$ give the respective probabilities of hitting two points located at a *fixed* distance from the 'interface', whereas the non-trivial scaling of the Brownian bridges on either side of the interface means that $\tilde p_\pm$ give the probabilities of hitting two points whose distances from the interface have the ratio C_+/C_-.

Most of the remainder of this section is devoted to the fact that:

Proposition 3.5. *The family of processes Y^ε given by (2.1) satisfies the assumptions of Theorem 3.2 with $\mathcal{L}_\pm = \frac{1}{2}C_\pm \partial_x^2$ and $p_\pm$ defined by the relations*

$$p_+ + p_- = 1\;, \qquad \frac{p_+}{p_-} = \frac{\lambda_+ C_+^2}{\lambda_- C_-^2}\;,$$

and $\lambda_\pm$ as in (1.4).

This yields the

Proof of Theorem 3.1. Before we start, let us remark that the initial condition y for the corrected process Y^ε and the initial condition x for the original process X^ε are related by $y = x + \varepsilon g(x/\varepsilon)$.

Note also that, thanks to the identity $\int_{t_0}^\infty e^{-\lambda s} F(s)\,ds = \int_{t_0}^\infty \lambda e^{-\lambda t} \int_{t_0}^y F(s)\,ds\,dt$ valid for any bounded measurable function F, the left hand side in (3.9) can be written as

$$\begin{aligned}\Delta(\varepsilon) &= \int_{t_0}^\infty \lambda e^{-\lambda t}\mathbb{E}_y\Big(f(Y^\varepsilon(t)) - f(Y^\varepsilon(t_0)) - \int_{t_0}^t Af(Y^\varepsilon(s))\,ds\,\Big|\,\mathscr{F}_{[0,t_0]}\Big)\,dt\\ &= \int_{t_0}^\infty \lambda e^{-\lambda t}\mathbb{E}_y\big(\mathcal{G}_f(Y^\varepsilon, t_0, t)\big|\,\mathscr{F}_{[0,t_0]}\big)\,dt\;.\end{aligned}$$

We have already established the weak precompactness of the family $\{\mathbb{P}^\varepsilon,\, \varepsilon > 0\}$ in the space $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$. The uniformity in x of the convergence of $\Delta(\varepsilon)$ to 0 then implies that for any n, any $0 \le t_1 < \cdots < t_n \le t_0$, and any bounded measurable function $G(x_1, \ldots, x_n)$, $x_i \in \mathbb{R}$,

$$\Big|\,\mathbb{E}_y\Big(\,G(Y^\varepsilon(t_1), \ldots, Y^\varepsilon(t_n))\cdot \int_{t_0}^\infty \lambda e^{-\lambda t}\mathcal{G}_f(Y^\varepsilon, t_0, t)\,dt\Big)\Big| \le \sup\big|G\big|\cdot\Delta(\varepsilon)\;. \tag{3.10}$$

If we furthermore assume that G is continuous, then the expression inside the expectation is a continuous function on $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$, so that any accumulation point X^0 satisfies

$$\int_{t_0}^\infty \lambda e^{-\lambda t}\mathbb{E}\Big(G\big(X^0(t_1), \ldots, X^0(t_n)\big)\mathcal{G}_f(X^0, t_0, t)\Big)\,dt = 0\;. \tag{3.11}$$

Since the integrand is a continuous function of t and a continuous function is determined uniquely by its Laplace transform, this implies that $\mathbb{E}\big(G(\ldots)\mathcal{G}_f(X^0, t_0, t)\big) =$

0 for all n and $0 \le t_1 < \cdots < t_n \le t_0$, so that in particular the random function $f(X^0(t)) - \int_0^t Af(X^0(s))\,ds$ is indeed a martingale in the filtration generated by the process X^0.

Since the laws of the starting points of X^ε are all equal to δ_x by construction, we conclude that the law of X^0 is indeed a solution of the martingale problem corresponding to A, starting from x_0. □

Proof of Proposition 3.5. The proofs of (3.6), (3.7) and (3.8) will be given as three separate propositions.

Proposition 3.6. *There exists $\tilde\eta > 0$ such that the process $Y^\varepsilon(t)$ satisfies (3.6) with $k(\varepsilon) = \varepsilon$, that is,*

$$\mathbb{E}_y\Big[e^{-\lambda\tau^\varepsilon} f\big(Y^\varepsilon(\tau^\varepsilon)\big) - f(x) + \int_0^{\tau^\varepsilon} e^{-\lambda t}\Big(\lambda f\big(Y^\varepsilon(t)\big) - \frac{1}{2}C_+^2 f''\big(Y^\varepsilon(t)\big)\Big)\,dt\Big] = \mathcal{O}\big(\varepsilon\big)\;,$$

for every function $f \in \mathcal{C}_0^\infty(\mathbb{R}_+)$, uniformly in $y \in [\tilde\eta\varepsilon, \infty)$, and similarly for the left side of the interface.

Proof. The treatment of both sides of the interface is identical, so we restrict ourselves to $\mathbb{R}_+$. As before, the initial condition y for the corrected process Y^ε and the initial condition x for the original process X^ε are related by $y = x + \varepsilon g(x/\varepsilon)$. Note that one has $g'_\pm(x) \neq -1$ for any x since otherwise, by uniqueness of the solutions to the ODE $g'' = -2(1+g')b$, this would entail that $g'_\pm(x) = -1$ over the whole interval $[0,1]$, in contradiction with the periodic boundary conditions.

By possibly making η slightly larger, we can (and will from now on) therefore assume that $g'(x) > -1$ uniformly over $x \in \mathbb{R}$, so that the correspondence $x \leftrightarrow y$ is a bijection. Since g is bounded, this shows that one can find $\tilde\eta > 0$ so that $y \notin [-\varepsilon\tilde\eta, \varepsilon\tilde\eta]$ implies that $x \notin [-\varepsilon\eta, \varepsilon\eta]$. In particular, fixing such a value for $\tilde\eta$ from now on, we see that the drift vanishes in the SDE satisfied by Y^ε, provided that we consider the process only up to time τ^ε.

Using the integration by parts formula and Itô's formula for each Y_y^ε we get

$$\begin{aligned} e^{-\lambda t} f\big(Y_y^\varepsilon(\tau^\varepsilon)\big) = f(x) &+ \int_0^{\tau^\varepsilon} e^{-\lambda s}\big(1 + g'_+(\varepsilon^{-1}X_x^\varepsilon(s))\big) f'\big(Y_y^\varepsilon(s)\big)\,dB_s \\ &- \int_0^{\tau^\varepsilon} e^{-\lambda s}\Big[\lambda f\big(Y_y^\varepsilon(s)\big) + \frac{1}{2}\big(1 + g'_+(\varepsilon^{-1}X_x^\varepsilon(s))\big)^2 f''\big(Y_y^\varepsilon(s)\big)\Big]\,ds \end{aligned} \tag{3.12}$$

Since the expectation of the stochastic integral vanishes (both f' and g'_+ are uniformly bounded), all that remains to be shown is that the last term in the above equation converges at rate ε to the same term with $(1+g')^2$ replaced by C_+^2.

This will be a consequence of the following result (variants of which are quite standard in the theory of periodic homogenization), which considers the fully periodic case. It is sufficient to consider this case in the situation at hand since we restrict ourselves to times before τ^ε, so that the process does not 'see' the interface.

Lemma 3.7. *Let $b\colon \mathbb{R}^n \to \mathbb{R}^n$ be smooth and periodic with fundamental domain $\Lambda \subset \mathbb{R}^n$, and denote by μ the (unique) probability measure on Λ invariant for the SDE*

$$dX(t) = b(X(t))\,dt + dB_t\;, \quad X(0) = x\;, \tag{3.13}$$

where B is a standard d-dimensional Wiener process. Assume furthermore that $\int_\Lambda b(x)\,\mu(dx) = 0$. (This condition will be referred to in the sequel as b being centred.)

Let $h\colon \mathbb{R}^d \to \mathbb{R}$ be any smooth function that is periodic with fundamental domain Λ and centred. Let furthermore $X^\varepsilon(t) = \varepsilon X(t/\varepsilon^2)$ and let τ_ε be a family of (possibly infinite) stopping times with respect to the natural filtration of B. Then, for every $F \in \mathcal{C}_0^4(\mathbb{R}^d, \mathbb{R})$ there exists $C > 0$ independent of τ_ε such that the bound

$$\mathbb{E}_x\left[\int_0^{\tau_\varepsilon} e^{-\lambda s} F\big(X^\varepsilon(s)\big) h\Big(\frac{X^\varepsilon(s)}{\varepsilon}\Big)\,ds\right] \le C\varepsilon\;,$$

holds for any $\varepsilon \in (0,1]$, uniformly in x.

Proof. Denote by $\mathcal{L} = \frac{1}{2}\Delta + \langle b(x), \nabla\rangle$ the generator of (3.13) and let g be the unique periodic centred solution to $\mathcal{L}g = h$. (Such a solution exists by the Fredholm alternative.) Applying Itô's formula to the process

$$\varepsilon^2 e^{-\lambda t} F\big(X^\varepsilon(t)\big) g\Big(\frac{X^\varepsilon(t)}{\varepsilon}\Big)\;,$$

we obtain the identity

$$\begin{aligned}
\varepsilon^2 e^{-\lambda \tau_\varepsilon} F\big(X^\varepsilon(\tau_\varepsilon)\big) g\Big(\frac{X^\varepsilon(\tau_\varepsilon)}{\varepsilon}\Big) &= \int_0^{\tau_\varepsilon} e^{-\lambda s} F\big(X^\varepsilon(s)\big) h\Big(\frac{X^\varepsilon(s)}{\varepsilon}\Big)\,ds \\
&\quad + \varepsilon^2 \int_0^{\tau_\varepsilon} -\lambda e^{-\lambda s} F\big(X(s)\big) g\Big(\frac{X(s)}{\varepsilon}\Big)\,ds \\
&\quad + \varepsilon \int_0^{\tau_\varepsilon} e^{-\lambda s} b\Big(\frac{X^\varepsilon(s)}{\varepsilon}\Big).\,\nabla F\big(X^\varepsilon(s)\big) g\Big(\frac{X^\varepsilon(s)}{\varepsilon}\Big)\,ds \\
&\quad + \frac{1}{2}\varepsilon^2 \int_0^{\tau_\varepsilon} e^{-\lambda s} \Delta F\big(X^\varepsilon(s)\big) g\Big(\frac{X^\varepsilon(s)}{\varepsilon}\Big)\,ds \\
&\quad + \varepsilon^2 \int_0^{\tau_\varepsilon} e^{-\lambda s} \nabla F\big(X^\varepsilon(s)\big) g\Big(\frac{X^\varepsilon(s)}{\varepsilon}\Big)\,dB_s \\
&\quad + \varepsilon \int_0^{\tau_\varepsilon} e^{-\lambda s} F\big(X^\varepsilon(s)\big) \nabla g\Big(\frac{X^\varepsilon(s)}{\varepsilon}\Big)\,dB_s \\
&\quad + \varepsilon \int_0^{\tau_\varepsilon} e^{-\lambda s} \Big(\nabla F\big(X^\varepsilon(s)\big).\,\nabla g\Big(\frac{X^\varepsilon(s)}{\varepsilon}\Big)\Big)\,ds\;.
\end{aligned}$$

The claim then follows by taking expectations and noting that all the functions of X^ε appearing in the various terms are uniformly bounded. □

Returning to the proof of Proposition 3.6, we first note that $f''(Y_y^\varepsilon) = f''(X_x^\varepsilon) + \mathcal{O}(\varepsilon)$, so that we can replace $f''(Y_y^\varepsilon)$ by $f''(X_x^\varepsilon)$ in (3.12), up to errors of $\mathcal{O}(\varepsilon)$. Applying Lemma 3.7 with $F = f''$ and $h = (1 + g_+')^2 - C_+^2$, the claim then follows at once. □

Proposition 3.8. *The convergence*

$$\mathbb{E}\left[\int_0^\infty e^{-\lambda t} 1_{(-\sqrt{\varepsilon},\sqrt{\varepsilon})}\big(Y_y^\varepsilon(t)\big)\,dt\right] \to 0 \tag{3.14}$$

takes place as $\varepsilon \to 0$, uniformly in the initial point $y \in \mathbb{R}$. In particular, (3.7) holds with $\delta(\varepsilon) = \sqrt{\varepsilon}$.

Proof. The main idea is to first perform a time-change that turns the diffusion coefficient of Y^ε into 1 and to then compare the resulting process to the process V^ε which is the solution to

$$dV^\varepsilon = b_V^\varepsilon(V^\varepsilon)\,dt + dB_t\;, \tag{3.15}$$

where the drift b_V^ε is given by

$$b_V^\varepsilon(x) = \begin{cases} -\frac{C_V}{\varepsilon} & \text{for } 0 \le x \le \hat\eta\varepsilon, \\ \frac{C_V}{\varepsilon} & \text{for } -\hat\eta\varepsilon \le x < 0, \\ 0 & \text{otherwise,} \end{cases} \tag{3.16}$$

for C_V and $\hat\eta$ some positive constants independent of ε to be determined below. An explicit resolvent equation then allows one to show that (3.14) with Y replaced by V tends to zero as $\varepsilon \to 0$, uniformly in the initial point.

First, let us start with the time change. As in the proof of Proposition 3.6, we assume that g is chosen in such a way that g' is bounded away (from below) from -1, so that there exists a constant $\phi > 0$ such that $g'(x) \ge \phi - 1$ for every $x \in \mathbb{R}$. In order to turn the diffusion coefficient of Y^ε into 1, we use the time change associated with the quadratic variation of the Y^ε,

$$\langle Y^\varepsilon, Y^\varepsilon\rangle(t) = \int_0^t \Big(1 + g'\Big(\frac{X^\varepsilon(s)}{\varepsilon}\Big)\Big)^2\,ds\;,$$

thus setting

$$C_t^\varepsilon = \inf\big\{t' > 0 : \langle Y^\varepsilon, Y^\varepsilon\rangle(t') > t\big\}\;.$$

Defining the function $\hat b = b + \mathcal{L}g$, the process $Z^\varepsilon(t) = Y^\varepsilon(C_t^\varepsilon)$ then satisfies the equation

$$Z^\varepsilon(t) = y + \int_0^{C_t^\varepsilon} \frac{1}{\varepsilon}\,\hat b\Big(\frac{X^\varepsilon(s)}{\varepsilon}\Big)\,ds + \int_0^{C_t^\varepsilon}\Big(1 + g'\Big(\frac{X^\varepsilon(s)}{\varepsilon}\Big)\Big)\,dB_s\;.$$

Note that the time-change was defined precisely in such a way that the second term in this expression is equal to some Brownian motion $B^\varepsilon(t)$. Inserting the expression for the time change, the first term can be rewritten as

$$\int_0^{C_t^\varepsilon} \frac{1}{\varepsilon}\,\hat b\Big(\frac{X^\varepsilon(s)}{\varepsilon}\Big)\,ds = \int_0^t \frac{1}{\varepsilon}\,\hat b\Big(\frac{X^\varepsilon(C_s^\varepsilon)}{\varepsilon}\Big)\Big(1 + g'\Big(\frac{X^\varepsilon(C_s^\varepsilon)}{\varepsilon}\Big)\Big)^{-2}\,ds\;.$$

It follows that the drift term is non-zero only when the time-changed process occupies the region $(-\varepsilon\eta - \varepsilon\|g\|_\infty,\, \varepsilon\eta + \varepsilon\|g\|_\infty)$ just as for the non time-changed process. To summarise, there exists a drift $\tilde b$ bounded uniformly by $\frac{C_V}{\varepsilon}$ for some constant $C_V > 0$ and vanishing outside of $(-\tilde\eta\varepsilon, \tilde\eta\varepsilon)$ for $\tilde\eta = \eta + \|g\|_\infty$, as well as a Brownian motion B^ε, so that the process Z_x^ε satisfies the SDE

$$dZ_x^\varepsilon = \tilde b(Z_x^\varepsilon)\,dt + dB_t^\varepsilon\;, \qquad Z_x^\varepsilon(0) = y\;. \tag{3.17}$$

Now, look at how the time change affects the expression (3.14), where we set $G^\varepsilon = (-\sqrt{\varepsilon}, \sqrt{\varepsilon})$:

$$\begin{aligned}\int_0^\infty e^{-\lambda t} 1_{G^\varepsilon}\big(Y^\varepsilon(t)\big)\,dt &= \int_0^{C_\infty^\varepsilon} e^{-\lambda t} 1_{G^\delta}\big(Y^\varepsilon(t)\big)\,dt \\ &= \int_0^\infty e^{-\lambda t} 1_{G^\varepsilon}\big(Z^\varepsilon(t)\big)\Big(1 + g'\Big(\frac{X^\varepsilon(C_t^\varepsilon)}{\varepsilon}\Big)\Big)^{-2}\,dt\end{aligned}$$

$$\leq \sup_{x\in\mathbb{R}}\big(1+g'(x)\big)^{-2}\int_0^\infty e^{-\lambda t}1_{G^\varepsilon}\big(Z^\varepsilon(t)\big)\,dt\;.$$

Hence if it can be shown that,

$$\mathbb{E}_y\left[\int_0^\infty e^{-\lambda t}1_{G^\varepsilon}\big(Z^\varepsilon(t)\big)\,dt\right]\to 0 \tag{3.18}$$

uniformly in the initial point x for the underlying process X^ε_x, as $\varepsilon\to 0$, then our claim follows. The idea is to bound (3.18) by the 'worst-case scenario' obtained by replacing the process Z^ε by the process V^ε described in (3.15).

One technical problem that arises is that it is tricky to get pathwise control on the behaviour of V due to the discontinuity of its diffusion coefficient. We therefore first compare Z^ε with the process U^ε_x solution to

$$dU^\varepsilon = b^\varepsilon_U(U^\varepsilon)\,dt + dB^\varepsilon_U(t)\;,\quad U^\varepsilon_x(0)=y\;, \tag{3.19}$$

where B^ε_U is a Brownian motion to be determined and b^ε_U is the Lipschitz continuous odd function defined on the positive real numbers by

$$b^\varepsilon_U(x)=\begin{cases}-\frac{C_V}{\varepsilon^2}\,x & \text{for } |x|\leq\varepsilon,\\ -\frac{C_V}{\varepsilon} & \text{for } \varepsilon<x\leq(2+\tilde\eta)\varepsilon,\\ -\frac{C_V}{\varepsilon^2}\,((3+\tilde\eta)\varepsilon-x) & \text{for } (2+\tilde\eta)\varepsilon<x\leq(3+\tilde\eta)\varepsilon,\\ 0 & \text{otherwise.}\end{cases}$$

The SDE (3.19) satisfies pathwise uniqueness, which is why we are using it as an intermediary between Z^ε and V^ε. Now, what we are going to do is, given a realisation of the Brownian motion B^ε driving Z^ε in (3.17), to choose the Brownian motion B^ε_U driving U^ε_x by changing the sign of the increments in such a way that the absolute value of U^ε_x is always less than or equal to $|Z^\varepsilon_x|+2\varepsilon$. By pathwise uniqueness, we are indeed free to choose the Brownian motion in (3.19). The choice of the Brownian motion is the content of the following lemma,

Lemma 3.9. *For every initial condition x, there exists a map $B^\varepsilon\mapsto B^\varepsilon_U$ that preserves Wiener measure and such that $|U^\varepsilon_x|\leq|Z^\varepsilon_x|+2\varepsilon$ almost surely. In particular, it follows that (3.18) is bounded by*

$$\mathbb{E}_x\Big(\int_0^\infty e^{-\lambda t}1_{(-\delta',\,\delta')}\big(U^\varepsilon_x(t)\big)\,dt\Big)\;, \tag{3.20}$$

where $\delta'=\delta+2\varepsilon$.

Proof. The construction works in the following way. Consider first the processes driven by the same realisation B^ε and define a stopping time τ_0 by $\tau_0=\inf\{t>0:|U^\varepsilon_x(t)|=|Z^\varepsilon_x(t)|+2\varepsilon\}$. This stopping time is strictly positive and one has $|U^\varepsilon_x(\tau_0)|\geq 2\varepsilon$. For times after τ_0, we determine B^ε_U by

$$B^\varepsilon_U(t)=B^\varepsilon_U(\tau_0)+sign\big(U^\varepsilon_x(\tau_0)\big)\int_{\tau_0}^t sign\big(Z^\varepsilon_x(s)\big)\,dB^\varepsilon(s)\;,$$

and we introduce the stopping time $\sigma_1=\inf\{t>\tau_0:|U^\varepsilon_x|=\varepsilon\}$. Since by construction U^ε does not change sign between τ_0 and σ_1, it then follows from the Itô-Tanaka formula that up to σ_1 one has

$$\begin{aligned} d|U^\varepsilon| &= b^\varepsilon_U\big(|U^\varepsilon|\big)\,dt + sign\big(Z^\varepsilon\big)\,dB^\varepsilon(t)\;,\\ d|Z^\varepsilon| &= sign\big(Z^\varepsilon\big)\tilde b^\varepsilon\big(Z^\varepsilon\big)\,dt + sign\big(Z^\varepsilon\big)\,dB^\varepsilon(t)+dL(t)\;,\end{aligned}$$

for some local time term L. Since the local time term always yields positive contributions and since it follows from the definition that $b_U^\varepsilon(u) \le sign(z)\tilde{b}^\varepsilon(z)$ for $|u| \ge \varepsilon$ and $|u| \le |z| + 2\varepsilon$, we can apply a simple comparison result for SDEs to conclude that the inequality $|U_x^\varepsilon| \le |Z_x^\varepsilon| + 2\varepsilon$ holds almost surely between times τ_0 and σ_0.

We then drive again both processes by the same noise and define as before τ_1 by $\tau_1 = \inf\{t > \sigma_1 : |U_x^\varepsilon(t)| = |Z_x^\varepsilon(t)| + 2\varepsilon\}$. Note that $\tau_1 > \sigma_1$ almost surely since one has $|U_x^\varepsilon(\sigma_0)| \le |Z_x^\varepsilon(\sigma_0)| + \varepsilon$. We then apply the previous construction iteratively, so that, setting $\sigma_0 = 0$, we have constructed B_U^ε by

$$B_U^\varepsilon(t) = \int_0^t \Big(\sum_{n=0}^\infty 1_{[\sigma_n,\tau_n)}(s) + \sum_{n=0}^\infty 1_{[\tau_n,\sigma_{n+1})}(s)\ sign\big(U_x^\varepsilon(\tau_n)Y_x^\varepsilon(s)\big)\Big)\, dB^\varepsilon(s)\ .$$

Since the process U^ε has finite quadratic variation and has to move by at least ε between any two successive stopping times, our sequence of stopping times does converge to infinity, so that $B_U^\varepsilon(t)$ is indeed a Brownian motion with the required property. □

In our next step, we compare the process U_x^ε that we just constructed with the process V_x^ε defined in (3.15), where we set $\hat{\eta} = 5 + \tilde{\eta}$. Since the drift coefficient is bounded, it follows from an application of Girsanov's theorem like in [RY91, Corollary IX.1.12] that this SDE has a solution for some Brownian motion B_V^ε, say.

We now fix B_V^ε and use it to construct a Brownian motion B_U^ε driving (3.19) in such a way that the absolute value of V_x^ε always stays less than $|U_x^\varepsilon| + 2\varepsilon$:

Lemma 3.10. *There exists a map $B_V^\varepsilon \mapsto B_U^\varepsilon$ that preserves Wiener measure and such that $|V_x^\varepsilon| \le |U_x^\varepsilon| + 2\varepsilon$ for all times almost surely. In particular, (3.20) is bounded by*

$$\mathbb{E}_x\Big(\int_0^\infty e^{-\lambda t} 1_{(-\delta'',\,\delta'')}(V_x^\varepsilon(t))\,dt\Big)\ , \tag{3.21}$$

with $\delta'' = \delta' + 2\varepsilon$.

Proof. The argument is virtually identical to that of Lemma 3.9, so we do not reproduce it here. □

It now remains to show that:

Lemma 3.11. *The expression (3.21) converges to* 0 *uniformly in the initial point as $\varepsilon \to 0$.*

Proof. We write ε for $5\varepsilon + \varepsilon\eta$ and δ for $\delta + 4\varepsilon$ for ease of notation, but this has no bearing on the rates of convergence of the aforementioned quantities and hence on the calculation. We have the identity

$$\begin{aligned}\mathbb{E}_x\Big[\int_0^\infty e^{-\lambda t} 1_{(-\delta,\,\delta)}\big(V_x^\varepsilon(t)\big)dt\Big] &= \int_0^\infty e^{-\lambda t} P_t^\varepsilon\big(1_{(-\delta,\,\delta)}\big)(x)\,dt \\ &= (\lambda - \mathcal{L}_V^\varepsilon)^{-1} 1_{(-\delta,\delta)}(x)\end{aligned} \tag{3.22}$$

by the resolvent equation (see for example [EK86, Chapter 1]), where $\mathcal{L}_V^\varepsilon$ is the generator of the Markov semigroup P_t^ε associated to V^ε.

We now proceed to computing this expression explicitly in order to show that its supremum tends to zero uniformly in x. In order to keep notations simple, we assume for the remainder of this proof that $\eta = C_V = 1$, which can always be achieved by rescaling space and redefining ε. In this case, the solution $f(x) = (\lambda - \mathcal{L}_V^\varepsilon)^{-1} 1_{(-\delta,\delta)}(x)$ to the resolvent equation can be assembled piecewise on the intervals $(-\infty,-\delta)$, $(-\delta,-\varepsilon)$, $(-\varepsilon,0)$, $(0,\varepsilon)$, (ε,δ) and (δ,∞) by making sure that it is $\mathcal{C}^1$ at each junction. Owing to the symmetry of the problem, the function f will be an even function of x, hence we only have to analyze it on one side of the origin.

The general solution on each interval can be written as

$$f(x) = \begin{cases} B_0 e^{-\sqrt{2\lambda}x} & \text{for } x \ge \delta, \\ \frac{1}{\lambda} + A_1 e^{\sqrt{2\lambda}x} + B_1 e^{-\sqrt{2\lambda}x} & \text{for } \varepsilon \le x \le \delta, \\ \frac{1}{\lambda} + \varepsilon^2 A_2 e^{\gamma_1 x} + B_2 e^{-\gamma_2 x} & \text{for } x \le \varepsilon, \end{cases}$$

where

$$\gamma_1 = \left(\frac{1}{\varepsilon^2} + 2\lambda\right)^{\frac{1}{2}} + \frac{1}{\varepsilon} = \frac{2}{\varepsilon} + \mathcal{O}(\varepsilon)$$

$$\gamma_2 = \left(\frac{1}{\varepsilon^2} + 2\lambda\right)^{\frac{1}{2}} - \frac{1}{\varepsilon} = \lambda\varepsilon + \mathcal{O}(\varepsilon^2)\;.$$

The reason for the somewhat strange choice of adding an explicit factor ε^2 in front of A_2 is justified *a posteriori* by noting that with this scaling, the matching conditions at ε and δ (as well as the fact that the derivative should vanish at the origin) yield the following linear system:

$$M \begin{pmatrix} B_0 \\ A_1 \\ B_1 \\ A_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{\lambda} \\ 0 \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & 0 & 0 & 2 & -\lambda \\ 0 & -1 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \end{pmatrix} + \mathcal{O}(\delta)\;.$$

To lowest order in ε and δ, this can easily be solved exactly, yielding

$$(B_0, A_1, B_1, A_2, B_2) = -\frac{1}{2\lambda}\big(0, 1, 1, \lambda, 2\big) + \mathcal{O}(\delta)\;.$$

Inserting this into the expression for f shows that $\sup_{x\in\mathbb{R}} |f(x)| = \mathcal{O}(\delta) = \mathcal{O}(\sqrt{\varepsilon})$, thus completing the proof. □

With the lemma regarding the resolvent calculation above, the proof of Proposition 3.8 is complete. □

We finally show that

Proposition 3.12. *For every $c > 0$, the exit probabilities from the interval $(-\delta,\delta)$ satisfy the bound*

$$\mathbb{P}_x\big[Y^\varepsilon(\sigma^\delta) \in I_i\big] = p_i + \mathcal{O}(\sqrt{\varepsilon})\;,$$

uniformly for $x \in [-c\varepsilon, c\varepsilon]$.

Proof. For the proof of this result, it turns out to be simpler to consider the original process $X^\varepsilon(t)$.

Whenever $Y^\varepsilon(t)$ exits the set $(-\delta,\delta)$, due to the deterministic relationship between the processes, $X^\varepsilon(t)$ exits a set $(-\delta',\delta'')$, where δ' and δ'' are contained in the interval $(\delta - \varepsilon\|g\|_\infty, \delta + \varepsilon\|g\|_\infty)$. Therefore, we just look at the exit of $X^\varepsilon(t)$ from an interval of this form as the computations are much easier to carry out.

This is due to the simpler form of the scale function for $X^\varepsilon(t)$ compared with that of $Y^\varepsilon(t)$. It follows from [RY91, Exercise VII.3.20] that the scale function of the diffusion on $\mathbb{R}$ with generator $\mathcal{L} = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}$ is given by,

$$s(x) = \int_c^x \exp\Big(-\int_c^y 2b(z)\sigma^{-2}(z)dz\Big)\,dy$$

where c is an arbitrary point in $\mathbb{R}$. Recall that the scale function of a real-valued process is a continuous, strictly increasing function such that for any $a < x < b$ in the set where the Markov process takes its values, one has

$$\mathbb{P}_x\big(T_b < T_a\big) = \frac{s(x)-s(a)}{s(b)-s(a)}\;,$$

where T_a, T_b are the first hitting times of the points a and b respectively. For $X^\varepsilon(t)$ we have that $\sigma = 1$ and the drift is equal to $\frac{1}{\varepsilon}\,b(x/\varepsilon)$. We are from now on going to use the notation $q_\varepsilon(y) = \big(1+\varepsilon g(\cdot/\varepsilon)\big)^{-1}(y)$ for the transformation that allows recovery of X^ε from Y^ε. We also denote by T_a the first hitting time of the point $a\in\mathbb{R}$ by the process X^ε. We also use the shorthand notation

$$F_\varepsilon(u) = \exp\Big(-2\int_0^u \frac{1}{\varepsilon}\,b(z/\varepsilon)\,dz\Big)\;,$$

so that the scale function for X^ε is given by $s(z) = \int_0^z F_\varepsilon(y)\,dy$.

With this notation at hand, we have, for $x\in(-\delta,\delta)$, that, denoting the escape time of $Y^\varepsilon(t)$ from $(-\delta,\delta)$ by σ^δ,

$$\begin{aligned}\mathbb{P}_x\big(Y^\varepsilon(\sigma^\delta)\in I_+\big) = \mathbb{P}_x\big(T_{\delta''} < T_{-\delta'}\big) &= \frac{s\big(q_\varepsilon(x)\big)-s(-\delta')}{s(\delta'')-s(-\delta')}\\ &= \frac{\int_0^{q_\varepsilon(x)} F_\varepsilon(y)\,dy - \int_0^{-\delta'} F_\varepsilon(y)\,dy}{\int_0^{-\delta''} F_\varepsilon(y)\,dy - \int_0^{-\delta'} F_\varepsilon(y)\,dy} = \frac{\int_{-\delta'}^{q_\varepsilon(x)} F_\varepsilon(y)\,dy}{\int_{-\delta'}^{\delta''} F_\varepsilon(y)\,dy}\;.\end{aligned}$$

Noting that F_ε has the scaling property $F_\varepsilon(u) = F_1(u/\varepsilon)$, we thus obtain the identity

$$\mathbb{P}_x\big(Y^\varepsilon(\sigma^\delta)\in I_+\big) = \frac{\int_{-\delta'}^{q_\varepsilon(x)} F_1(y/\varepsilon)\,dy}{\int_{-\delta'}^{\delta''} F_1(y/\varepsilon)\,dy} = \frac{\int_{-\delta/\varepsilon}^{0} F_1(y)\,dy + \mathcal{O}(1)}{\int_{-\delta/\varepsilon}^{\delta/\varepsilon} F_1(y)\,dy + \mathcal{O}(1)}\;, \tag{3.23}$$

where we used the fact that $q_\varepsilon(x) = \mathcal{O}(\varepsilon)$ and F_1 is uniformly bounded, due to the fact that the functions $b_\pm$ are centred by assumption.

Note now that the effective diffusion coefficients $C_\pm$ can alternatively be expressed as [PS08, Sec 13.6]

$$C_+^2 = 2\Big[\int_\eta^{\eta+1}\exp\big(-2V(u)\big)\,du\;\int_\eta^{\eta+1}\exp\big(2V(u)\big)\,du\Big]^{-1}\;,$$

and similarly for C_-. Therefore, since F_1 is periodic away from $[-\eta,\eta]$, it follows immediately from the definitions of $\lambda_\pm$ and $C_\pm$ that

$$\int_{-N}^0 F_1(y)\,dy = \frac{1}{2C_-^2\lambda_-}N + \mathcal{O}(1)\;,\qquad \int_0^N F_1(y)\,dy = \frac{1}{2C_+^2\lambda_+}N + \mathcal{O}(1)\;.$$

Since $\delta = \sqrt{\varepsilon}$, combining these bounds with (3.23) implies that $\mathbb{P}_x\big(Y^\varepsilon(\sigma^\delta)\in I_+\big) = C_+^2\lambda_+/(C_-^2\lambda_+ + C_+^2\lambda_-) + \mathcal{O}(\sqrt{\varepsilon})$, from which the requested bound follows. $\square$

Combining Propositions 3.6, 3.8, and 3.12 completes the proof of Proposition 3.5. □

4. Uniqueness and characterization of the relevant martingale problem

For completeness, we show in this section is to show that the martingale problem associated to the operator A defined in Theorem 3.1 has a unique solution. This is a special case of a classical result for diffusions on a graph, see [FW98, Theorem 2.1] or Feller's original works [Fel52, Fel57], but we show here how to obtain it in a straightforward way as a byproduct of a general abstract result by Ethier and Kurtz [EK86]. Recall that the domain $\mathscr{D}(A)$ consists of functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $f|_{I_\pm} \in \mathcal{C}_0^\infty(I_\pm)$ and such that $p_+ f'(0^+) = p_- f'(0^-)$. For definiteness, we furthermore define $(Af)(0) = \frac{1}{2}\big(\mathcal{L}_+ f(0^+) + \mathcal{L}_- f(0^-)\big)$. (These limits exist by the assumptions on f.)

Well-posedness of the martingale problem can then be obtained as a consequence of [EK86, Theorem IV.4.1] which we recall here (linear operators are identified with their graphs):

Theorem 4.1. *Let E be a separable metric space, and let $A \subset \mathcal{B}_b(E) \times \mathcal{B}_b(E)$ be linear and dissipative (here we denote by $\mathcal{B}_b(E)$ the space of bounded Borel measurable functions on E). Suppose there exists a linear operator $A' \subset A$ such that $\overline{\mathscr{R}(\lambda - A')} = \overline{\mathscr{D}(A')} \equiv M$ for some $\lambda > 0$, and $M \subset \mathcal{B}_b(E)$ is separating. Let $\mu \in \mathscr{P}(E)$ and suppose X is a solution of the martingale problem for (A, μ). Then X is a Markov process corresponding to the semigroup on M generated by the closure of A', and uniqueness holds for the martingale problem for (A, μ).*

It will follow that:

Theorem 4.2. *The solution to the martingale problem corresponding to A given at the beginning of the section is unique.*

Proof. In order to use Theorem 4.1, we need to define the appropriate quantities in the theorem, and to show that the aformentioned martingale problem has a solution, which will be shown later using a rescaled skew Brownian motion. We set $\mathscr{D}(A')$ to be the subset of $\mathscr{D}(A)$ consisting of functions that are smooth, vanish at infinity together with all their derivatives and that are such that Af is continuous.

Let us start by showing that A is dissipative, namely that $\|\lambda f - Af\| \geq \lambda \|f\|$ for every $f \in \mathscr{D}(A)$ and every $\lambda > 0$. Here, the norm is the supremum norm so that, if the point x_1 at which the maximum value of the modulus of f occurs is a non-zero maximum then since L_i are just $\frac{d^2}{dx^2}$ multiplied by some strictly positive coefficient we have that $\lambda f(x_1) - \frac{1}{2} C_\pm \frac{d^2 f}{dx^2}(x_1) \geq \lambda f(x_1)$ since $\frac{d^2 f}{dx^2}(x_1) \geq 0$. This implies that $\|\lambda f - Af\| \geq \lambda \|f\|$. If the point at which the maximum modulus obtained is a minimum then similar reasoning applies. So the only case that could still possibly invalidate the dissipativity of A is the case where the maximum of $|f|$ occurs at 0. Since we know that the gradients on either side of the origin are strictly positive multiples of one another, a maximum at zero implies that the limit of both gradients is zero, so that one would have again $\lim_{x\to 0^\pm} \frac{d^2 f}{dx^2}(x_1) \leq 0$, thus allowing to apply the same reasoning as before and concluding that A is indeed dissipative.

It is easy to see that $M := \overline{\mathscr{D}(A')} = \mathcal{C}_0(\mathbb{R})$, the space of continuous functions vanishing at infinity, which is separating, so that it only remains to show that $\overline{\mathscr{R}(\lambda - A')} = \mathcal{C}_0(\mathbb{R})$ for some $\lambda > 0$. We therefore fix $f_1 \in \mathscr{D}(A')$, f_1 compactly supported, and seek to construct $f_2 \in \mathscr{D}(A')$ such that $Af_2 = f_1$. The obvious starting point for constructing such an f_2 is to take any two solutions to the differential equation $\lambda f_\pm - \frac{1}{2}C_\pm^2 f_\pm'' = f_1$ on both $(-\infty, 0]$ and $[0, \infty)$ respectively and to try to match them at 0. Since f_1 vanishes outside a compact set, outside this set on either side of the origin we must have,

$$f_+ = A_+ e^{-K_+ x} , \qquad f_- = A_- e^{K_- x} ,$$

for some positive constants $K_\pm$ such that $\frac{1}{2}C_\pm^2 K_\pm^2 = \lambda$. The general solution vanishing at ∞ is therefore given by

$$f_+ = f_+^0 + A_+ e^{-K_+ x} , \qquad f_- = f_-^0 + A_- e^{K_- x} ,$$

where the $f_\pm^0$ are the unique compactly supported solutions to $(\lambda - \mathcal{L}_\pm) f_\pm^0 = f_1$.

it remains to show that it is possible to choose the constants $A_\pm$ in such a way that the function f_2 defined by $f_2(x) = f_\pm(x)$ for $x \in \mathbb{R}_\pm$ is continuous (one constraint) and has derivatives satisfying $p_+ f_+'(0^+) = p_- f_-'(0^-)$. The linear system of equations to be satisfied is given by:

$$\begin{pmatrix} 1 & 1 \\ -K_+ & K_- \end{pmatrix} \begin{pmatrix} A_+ \\ A_- \end{pmatrix} = \begin{pmatrix} f_-^0(0) - f_+^0(0) \\ p_-(f_-^0)'(0) - p_+(f_+^0)'(0) \end{pmatrix} .$$

Since the constants $K_\pm$ are strictly positive, the determinant of this matrix is positive, so that the system has a unique solution, thus concluding the proof of Theorem 4.2 by applying Theorem 4.1. □

We finally turn to the characterization of solutions to the martingale problem corresponding to A as instances of skewed Brownian motion as constructed in the introduction.

The skew Brownian motion B_p of 'skewness' parameter p is known to have generator $\mathcal{L}_p = \frac{1}{2}\frac{d^2}{dx^2}$ on the set of functions that are continuous, twice continuously differentiable except at the origin where we have $pf'(0^+) = (1-p)f'(0^-)$, see for example the review article [Lej06]. The process $B_{C_\pm,p}$ constructed in the introduction is given by

$$B_{C_\pm,p}(t) = G(B_p(t)) , \qquad G(x) = \begin{cases} C_+ x & \text{if } x \geq 0, \\ C_- x & \text{otherwise.} \end{cases}$$

Since G is a continuous bijection, this is again a strong Markov process and it has generator given by $Af = \big(\mathcal{L}_p(f \circ G)\big) \circ G^{-1}$ with $\mathscr{D}(A) = \{f \,:\, f \circ G \in \mathscr{D}(\mathcal{L}_p)\}$. Using the relation (1.3), it is now a straightforward calculation to show that $\mathscr{D}(A)$ consists precisely of those functions satisfying the derivative condition in (3.4).

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