Modulation Equations:
Stochastic Bifurcation in Large Domains

November 23, 2004

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Abstract

We consider the stochastic Swift-Hohenberg equation on a large domain near its change of stability. We show that, under the appropriate scaling, its solutions can be approximated by a periodic wave, which is modulated by the solutions to a stochastic Ginzburg-Landau equation. We then proceed to show that this approximation also extends to the invariant measures of these equations.

Contents

1 Introduction

We present a rigorous approximation result of stochastic partial differential equations (SPDEs) by amplitude equations near a change of stability. In order to keep notations at a bearable level, we focus on approximating the stochastic Swift-Hohenberg equation by the stochastic Ginzburg-Landau equation, although our results apply to a larger class of stochastic PDEs or systems of SPDEs. Similar results are well-known in the deterministic case, see for instance [CE90, MSZ00]. However, there seems to be a lack of theory when noise is introduced into the system. In particular, the treatment of extended systems (*i.e.* when the spatial variable takes values in an unbounded domain) is still out of reach of current techniques.

In a series of recent articles [BMPS01, Blö03a, Blö03b, BH04], the amplitude of the dominating pattern was approximated by a stochastic ordinary differential equation (SODE). On a formal level or without the presence of noise, the derivation of these results is well-known, see for instance (4.31) or (5.11) in the comprehensive review article [CH93] and references therein. This approach shows its limitations on large domains, where the spectral gap between the dominating pattern and the rest of the equation becomes small. It is in particular not appropriate to explain modulated pattern occurring in many physical models and experiments (see *e.g.* [Lyt96, LM99] or [CH93] for a review). The validity of the SODE-approximation is limited to a small neighbourhood of the stability change, which shrinks, as the size of the domain gets large.

For deterministic PDEs on unbounded domains it is well-known, see *e.g.* [CE90, MS95, KSM92, Sch96], that the dynamics of the slow modulations of the pattern can be described by a PDE which turns out to be of Ginzburg-Landau type.

Since the theory of translational invariant SPDEs on unbounded domains is still far from being fully developed, we adopt in the present article a somewhat intermediate approach, considering large but bounded domains in order to avoid the technical difficulties arising for SPDEs on unbounded domains. Note that the same approach has been used in [MSZ00] to study the deterministic Swift-Hohenberg equation. It does not seem possible to adapt the deterministic theory directly to the stochastic equation. One major obstacle is that the whole theory for deterministic PDE relies heavily on good a-priori bounds for the solutions of the amplitude equation in spaces of sufficiently smooth functions. Such bounds are unrealistic for our stochastic amplitude equation, since it turns out to be driven by space-time white noise. Its solutions are therefore only α-Hölder continuous in space and time for $\alpha < 1/2$. Nevertheless, the choice of large but bounded domains captures and describes all the essential features of how noise in the original equation enters the amplitude equation.

1.1 Setting and results

In this article, we concentrate on deriving the stochastic Ginzburg-Landau equation as an amplitude equation for the stochastic Swift-Hohenberg equation, though we expect that similar results hold for a much wider class of equations, see remark 2.5. The Swift-Hohenberg equation is a celebrated toy model for the convective instability in the Rayleigh-Bénard convection. A formal derivation of the equation from the Boussinesq approximation of fluid dynamics can be found in [HS77].

In the following we consider solutions to

$$\partial_t U = -(1+\partial_x^2)^2 U + \varepsilon^2 \nu U - U^3 + \varepsilon^{\frac{3}{2}} \xi_\varepsilon \tag{SH}$$

where $U(x,t) \in \mathbf{R}$ satisfies periodic boundary conditions on $D_\varepsilon = [-L/\varepsilon, L/\varepsilon]$. The noise ξ_ε is assumed to be real-valued homogeneous space-time noise. To be more precise ξ_ε is a distribution-valued centred Gaussian field such that

$$\mathbb{E}\xi_\varepsilon(x,s)\xi_\varepsilon(y,t) = \delta(t-s)q_\varepsilon(|x-y|)\;. \tag{1.1}$$

The family of correlation functions q_ε is assumed to converge in a suitable sense to a correlation function q. One should think for the moment of q_ε as simply being the $2L/\varepsilon$-periodic continuation of the restriction of q to D_ε. We will state in Assumption 7.4 the precise assumptions on q and q_ε. This will include space-time white noise and noise with bounded correlation length.

Before we formulate our main results, let us briefly discuss why we expect (SH) to have a scaling limit of the form

$$U(x,t) = 2\varepsilon \mathrm{Re}(a(\varepsilon x, \varepsilon^2 t)e^{ix})\;, \tag{1.2}$$

for small values of ε and why the factor $\varepsilon^{\frac{3}{2}}$ in front of the noise in equation (SH) is the correct factor to balance with the linear term $\varepsilon^2\nu U$ and the nonlinearity U^3 so that all three contribute to the limiting equation, eqn. (1.4) below. Since the nonlinearity dominates the linear instability at $U \gg \varepsilon$, we expect the solutions to (SH) to be of order ε, hence the term ε in front of the right-hand side of (1.2). It is then natural to consider timescales of order ε^{-2}, so that both the linear instability and the nonlinearity contribute significantly to the dynamics. This explains the argument $\varepsilon^2 t$. Concerning the relevant spacescale and the term e^{ix}, note that if U is "demodulated" by writing it as $U(x,t) = \mathrm{Re}(A(x,t)e^{ix})$, then the differential operator $-(1+\partial_x^2)^2$ acting on U is close to a multiple of the Laplacian acting on A (neglecting terms of order $\partial_x^3 A$ and $\partial_x^4 A$). This suggests that one should look at the solutions on a spacescale of ε^{-1} (since then $\partial_x^2 A \approx \varepsilon^2 A$ is of the same order as the linear instability and the nonlinearity), if one wants the linear differential operator to give a non-trivial contribution in the scaling limit. It remains to explain the factor $\varepsilon^{\frac{3}{2}}$ in front of the noise. This is an immediate consequence of a dimensional analysis of the stochastic heat equation

$$\partial_t A = \partial_x^2 A + J\,\xi\;, \tag{1.3}$$

(where ξ is space-time white noise and J is the noise strength), which is expected to describe the scaling limit of (SH) if $\nu = 0$ and no nonlinearity is present. The scaling behaviour of ξ, formally given by $\xi(\alpha x, \beta t) \stackrel{\text{law}}{=} (\alpha\beta)^{-\frac{1}{2}}\,\xi(x,t)$ immediately implies that on a space interval of order ε^{-1} and a time interval of order ε^{-2}, solutions to (1.3) are of order $J\varepsilon^{-\frac{1}{2}}$. Therefore, the noise should enter into (SH) with a prefactor of order $J \approx \varepsilon^{\frac{3}{2}}$, so that the corresponding contribution on the time and space scales under consideration is of order ε. Another way of seeing this is to notice that the solutions to the stochastic heat equation are (almost) $\frac{1}{4}$-Hölder continuous in time and $\frac{1}{2}$-Hölder continuous in space. This roughness in time is a direct consequence of the singularity of order $t^{-\frac{1}{4}}$ in the L^2-norm of the Heat kernel (see *e.g.* [DPZ96, Thm 5.20]). Therefore, one would expect their size to be of order $(t^{\frac{1}{4}} + x^{\frac{1}{2}})J$. On the time and space scales under consideration, we see again that $J \approx \varepsilon^{\frac{3}{2}}$ results in a contribution of order ε. Note that if we were to study the Swift–Hohenberg equation in a bounded domain D not scaling with ε, then a noise strength of ε^2 would lead to the correct scaling, cf. [BMPS01].

The main result of this article is an approximation result for solutions to (SH) by means of solutions to the stochastic Ginzburg-Landau equation. We consider a class of "admissible" initial conditions given in Definition 3.4 below. This class is slightly larger than that of $\mathcal{H}^1$-valued random variables with bounded moments of all orders and is natural for the problem at hand, due to the lack of uniform $\mathcal{H}^1$–estimates for the stochastic convolution. We show in Theorem 5.1 that the solution of (SH) with arbitrary initial conditions becomes admissible after a transient time.

Our main result (cf. Theorem 4.1) is the following:

Theorem 1.1 (Approximation) *Let U be given by the solution of (SH) with an admissible initial condition written as $U_0(x) = 2\varepsilon\mathrm{Re}(a_0(\varepsilon x)e^{ix})$. Consider the solution $a(X,T)$ to the stochastic Ginzburg-Landau equation*

$$\partial_T a = 4\partial_X^2 a + \nu a - 3|a|^2 a + \sqrt{\hat{q}(1)}\,\eta\;, \quad X \in [-L, L]\;, a(0) = a_0\;, \qquad (1.4)$$

where η is complex space-time white noise and $\hat{q}$ denotes the Fourier transform of q. Here, a is subject to suitable boundary conditions, i.e. those boundary conditions such that $a(X,T)e^{iX/\varepsilon}$ is $2L$-periodic. Then, for every $T_0 > 0$, $\kappa > 0$, and $p \geq 1$, one can find joint realisations of the noises η and ξ_ε such that

$$\Big(\mathbb{E} \sup_{\varepsilon^2 t \in [0,T_0]} \sup_{x \in D_\varepsilon} |U(x,t) - 2\varepsilon\mathrm{Re}(a(\varepsilon x, \varepsilon^2 t)e^{ix})|^p\Big)^{1/p} \leq C_{\kappa,p}\,\varepsilon^{3/2-\kappa} \qquad (1.5)$$

for every $\varepsilon \in (0,1]$.

Note that solutions to (SH) tend to be of order ε, as can be seen from the fact that this is the point where the dissipative nonlinearity starts to dominate the linear instability. Therefore, the ratio between the size of the error and the size of the solutions is of order $\varepsilon^{1/2}$. Using an argument similar to the one in [BH04], it is then straightforward to obtain an approximation result on the invariant measures for (SH) and (1.4):

Theorem 1.2 (Invariant Measures) *Let $\nu_{\star,\varepsilon}$ be the invariant measure for (1.4) and let $\mu_{\star,\varepsilon}$ be an invariant measure for (SH). Then, one can construct random variables $a_\star$ and $U_\star$ with respective laws $\nu_{\star,\varepsilon}$ and $\mu_{\star,\varepsilon}$ such that for every $\kappa > 0$ and $p \geq 1$*

$$\Big(\mathbb{E} \sup_{x \in D_\varepsilon} |U_\star(x) - 2\varepsilon\mathrm{Re}(a_\star(\varepsilon x)e^{ix})|^p\Big)^{1/p} \leq C_{\kappa,p}\,\varepsilon^{3/2-\kappa}\;,$$

for every $\varepsilon \in (0,1]$.

Let us remark that $\nu_{\star,\varepsilon}$ is actually independent of ε, provided $L \in \varepsilon\pi\mathbf{N}$.

Remark 1.3 The correction $\varepsilon^{-\kappa}$ appearing in Theorems 1.1 and 1.2 is a direct consequence of the error estimates on the linearised equations obtained in Section 7. One could in principle obtain logarithmic bounds using the Fernique-Talagrand theorem from the theory of Gaussian processes. It is not expected, however, that a bound $\mathcal{O}(\varepsilon^{3/2})$ without any corrections holds.

Most of the present article is devoted to the proof of Theorem 1.1. We will then prove attractivity, Theorem 5.1 in Section 5 and Theorem 1.2 in Section 6, while Section 7 provides a very general approximation result for linear equations, that is used in the proof of Theorem 1.1.

The remainder of this paper is organised as follows. Section 2 is devoted to a formal justification of our results. The main step in the proof of Theorem 1.1 is then to define a *residual*, which measures how well a given process approximates solutions to (SH) via the variation of constants formula. Section 3 provides estimates for this residual that are used in Section 4 to prove the main approximation result. Section 5 justifies the assumptions on the initial conditions required for the proof of the approximation result, and Section 6 applies the result to the approximation of invariant measures. The final Section 7 provides the approximation result for linear equations in a fairly general setting.

Acknowledgements

We are grateful to the anonymous referee for his constructive criticism of an earlier version of this paper.

2 Formal Derivation of the Main Result

In order to simplify notations, we work from now on with the rescaled version $u(x,t)$ of the solutions of (SH), defined through $U(x,t) = \varepsilon u(\varepsilon x, \varepsilon^2 t)$. Then, u satisfies the equation

$$\partial_t u = -\varepsilon^{-2}(1+\varepsilon^2\partial_x^2)^2 u + \nu u - u^3 + \tilde{\xi}_\varepsilon \;, \tag{2.1}$$

with periodic boundary conditions on the domain $[-L, L]$. Here, we defined the rescaled noise $\tilde{\xi}_\varepsilon(x,t) = \varepsilon^{-3/2}\xi_\varepsilon(\varepsilon^{-1}x, \varepsilon^{-2}t)$. This is obviously a real-valued Gaussian noise with covariance given by

$$\mathbb{E}\tilde{\xi}_\varepsilon(x,t)\tilde{\xi}_\varepsilon(y,s) = \delta(t-s)\varepsilon^{-1}q_\varepsilon(\varepsilon^{-1}|x-y|) \;.$$

We define the operator $\mathcal{L}_\varepsilon = -1 - \varepsilon^{-2}(1+\varepsilon^2\partial_x^2)^2$ subject to periodic boundary conditions on $[-L, L]$ and we set $\tilde{\nu} = 1 + \nu$, so that (2.1) can be rewritten as

$$\partial_t u = \mathcal{L}_\varepsilon u + \tilde{\nu} u - u^3 + \tilde{\xi}_\varepsilon \;. \tag{SH$_\varepsilon$}$$

In order to handle the fact that the dominating modes $e^{\pm ix/\varepsilon}$ are not necessarily $2L$-periodic, we introduce the quantities

$$N_\varepsilon = \Big[\frac{L}{\varepsilon\pi}\Big]\;,\quad \delta_\varepsilon = \frac{1}{\varepsilon} - \frac{\pi}{L}N_\varepsilon\;,\quad \varrho_\varepsilon = N_\varepsilon \frac{\pi\varepsilon}{L}\;,$$

where $[\,x\,] \in \mathbf{Z}$ is used to denote the nearest integer of a real number x with the conventions that $[\frac{1}{2}] = \frac{1}{2}$ and $[-x] = -[x]$.

With these notations, we rewrite the amplitude equation in a slightly different way. Setting $A(x,t) = a(x,t)e^{i\delta_\varepsilon x}$, (1.4) is equivalent to

$$\partial_t A = \Delta_\varepsilon A + \tilde\nu A - 3|A|^2 A + \sqrt{\hat q(1)}\eta\;,\quad \Delta_\varepsilon = -1 - 4(i\partial_x + \delta_\varepsilon)^2\;, \tag{GL}$$

with *periodic* boundary conditions, where η is another version of complex space-time white noise. This transformation is purely for convenience, since periodic boundary conditions are more familiar.

Remark 2.1 Note that the limiting equation (GL) does still depend on ε through δ_ε. This effect is a consequence of the fact that our domain is large but nevertheless bounded and was already noticed in [MSZ00]. It is obvious however that the "drift" term $2i\delta_\varepsilon\partial_x$ in (GL) vanishes if we choose to let $\varepsilon \to 0$ along the sequence $L/(\pi\varepsilon) \in \mathbf{N}$. Note furthermore that $|\delta_\varepsilon|$ is bounded by $\frac{\pi}{2L}$ independently of ε. As far as bounds are concerned, the reader is therefore encouraged to think of (GL) as being independent of ε and to think of δ_ε as being 0.

Before we proceed further, we fix a few notations that will be used throughout this paper. We will consider solutions to (SH_ε) and (GL) in various function spaces, but let us for the moment consider them in $\mathrm{L}^2([-L,L])$. We thus denote by $\mathcal{H}_u$ the L^2-space of real-valued functions on $[-L,L]$ which will contain the solutions to (SH_ε) and by $\mathcal{H}_a$ the L^2-space of complex-valued functions on $[-L,L]$ which will contain the solutions to (GL). We define the norm in $\mathcal{H}_u$ as half of the usual L^2-norm, *i.e.*

$$\|u\|_u^2 = \frac{1}{2}\int_{-L}^{L} u^2(x)\,dx\;,\quad \|A\|_a^2 = \int_{-L}^{L} |A(x)|^2\,dx\;, \tag{2.2}$$

for all $u \in \mathcal{H}_u$ and all $A \in \mathcal{H}_a$.

Remark 2.2 The choice of adding a factor $\frac{1}{2}$ in $\|\cdot\|_u$ may seem unusual and confusing. However, this is the only way of making the operators π_ε and ι_ε defined in (2.3) and (2.4) below a projection and an isometric embedding respectively. The reason for not changing (2.3) and (2.4) instead is one of legacy: this is indeed the notation used throughout all the existing literature. If we were to remove the factor 2 in (2.3), the term $a|a|^2$ in (1.4) would have a prefactor 12 instead of 3, thus clashing with the existing literature on the subject.

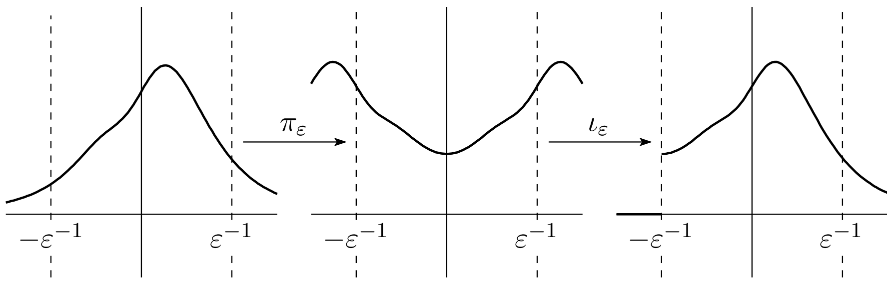

Figure 1: Action of π_ε and ι_ε.

We introduce the projection $\pi_\varepsilon : \mathcal{H}_a \to \mathcal{H}_u$ used in (1.5), *i.e.*

$$(\pi_\varepsilon A)(x) = 2\mathrm{Re}\big(A(x)e^{i\pi N_\varepsilon x/L}\big) \; . \tag{2.3}$$

We also define the injection $\iota_\varepsilon : \mathcal{H}_u \to \mathcal{H}_a$ by

$$(\iota_\varepsilon u)(x) = u_+ \exp(-i\pi N_\varepsilon x/L) \; , \tag{2.4}$$

where, for $u = \sum_{k\in\mathbf{Z}} u_k \exp(i\pi k/L)$, we defined $u_+ = \sum_{k>0} u_k \exp(i\pi k/L) + \frac{1}{2}u_0$. Since u is real-valued, one has of course the equality $u = u_+ + \overline{u_+}$, where $\overline{u_+}$ denotes the complex conjugate of u_+. Furthermore, one has the relations

$$\pi_\varepsilon \circ \iota_\varepsilon = \iota_\varepsilon^* \circ \iota_\varepsilon = \mathrm{Id} \; , \tag{2.5}$$

and the embedding ι_ε is isometric. Here, $\iota_\varepsilon^* : \mathcal{H}_u \to \mathcal{H}_a$ denotes the adjoint of ι_ε. We also define the space $\mathcal{H}_\iota \subset \mathcal{H}_a$ as the image of ι_ε. Equation (2.5) implies in particular that $\pi_\varepsilon = \iota_\varepsilon^*$, if both operators are restricted to $\mathcal{H}_\iota$. Note also that ι_ε is *not* a bounded operator between the corresponding L^∞ spaces, even though π_ε is.

Remark 2.3 Intuitively, the action of π_ε in Fourier space is to first translate the spectrum to the right by ε^{-1} and then to add its reflection around the $k = 0$ axis. The effect of ι_ε is to first cut off the $k < 0$ part and then translate the rest to the left by ε^{-1}. Figure 2 illustrates the successive actions of π_ε and ι_ε on an arbitrary function in Fourier space.

With these notations in mind, we give a formal argument that shows why (GL) is expected to yield a good approximation for (SH_ε). First of all, note that even though $\iota_\varepsilon \circ \pi_\varepsilon$ is not the identity, it is close to the identity when applied to a function which is such that its Fourier modes with wavenumber larger than ε^{-1} are small. This is indeed expected to be the case for the solutions A to (GL), since the heat semigroup $e^{\Delta_\varepsilon t}$ strongly damps high frequencies.

Hence, $\iota_\varepsilon \pi_\varepsilon A \approx A$. Therefore, making the ansatz $u = \pi_\varepsilon A$ and plugging it into (SH_ε) yields

$$\partial_t A \approx \iota_\varepsilon \mathcal{L}_\varepsilon \pi_\varepsilon A + \tilde{\nu} A - \iota_\varepsilon (\pi_\varepsilon A)^3 + \iota_\varepsilon \tilde{\xi}_\varepsilon \; .$$

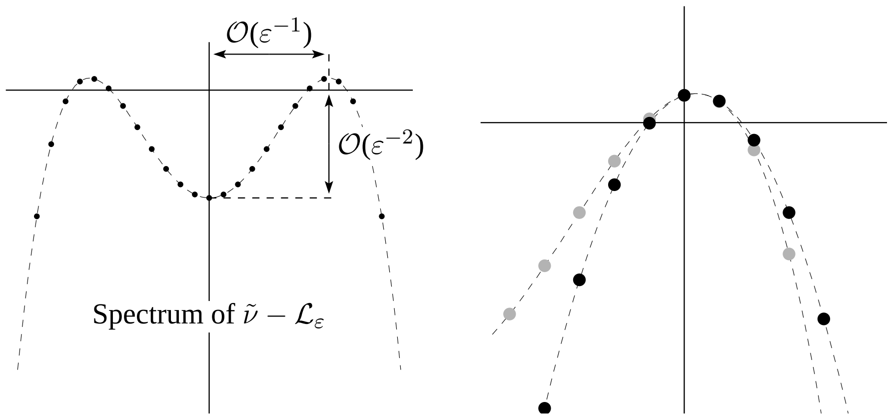

Figure 2: Spectra of the linear parts.

The left part of Figure 2 shows the spectrum of $\tilde{\nu} + \mathcal{L}_\varepsilon$. The right part shows the spectrum of $\iota_\varepsilon(\tilde{\nu} + \mathcal{L}_\varepsilon)\pi_\varepsilon$ (which is interpreted as a self-adjoint operator from $\mathcal{H}_\iota$ to $\mathcal{H}_\iota$) in grey and the spectrum of $\Delta_\varepsilon + \tilde{\nu}$ in black. One sees that the two are becoming increasingly similar as $\varepsilon \to 0$, since the tip of the curve becomes increasingly well approximated by a parabola.

Expanding the term $(\pi_\varepsilon A)^3$ we get

$$(\pi_\varepsilon A)^3 = A^3 e^{3i\pi N_\varepsilon x/L} + 3A|A|^2 e^{i\pi N_\varepsilon x/L} + 3\bar{A}|A|^2 e^{-i\pi N_\varepsilon x/L} + \bar{A}^3 e^{-3i\pi N_\varepsilon x/L} \;.$$

Therefore, one has

$$\iota_\varepsilon(\pi_\varepsilon A)^3 \approx A^3 e^{3i\pi N_\varepsilon x/L} + 3A|A|^2 \;.$$

Since the term with high wavenumbers will be suppressed by the linear part, we can arguably approximate this by $3A|A|^2$, so that we have

$$\partial_t A \approx \Delta_\varepsilon A + \tilde{\nu} A - 3|A|^2 A + \iota_\varepsilon \tilde{\xi}_\varepsilon \;. \tag{2.6}$$

It remains to analyse the behaviour of $\iota_\varepsilon\tilde{\xi}_\varepsilon$ in the limit of small values of ε. Note that we can expand $\tilde{\xi}_\varepsilon$ in Fourier series, so that

$$\tilde{\xi}_\varepsilon(x,t) \stackrel{\text{law}}{=} c_L \sum_{k\in\mathbf{Z}} \sqrt{\hat{q}_\varepsilon(\varepsilon k\pi/L)}\xi_k(t) e^{ik\pi x/L} \;,$$

where the $\xi_k(t)$ denote complex independent white noises, with the restriction that $\xi_{-k} = \overline{\xi_k}$, and where we set $c_L = 1/\sqrt{2L}$. On a formal level, this yields for $\iota_\varepsilon\tilde{\xi}_\varepsilon$

$$\begin{aligned}\iota_\varepsilon\tilde{\xi}_\varepsilon(x,t) &\stackrel{\text{law}}{\approx} \sum_{k=0}^\infty c_L \sqrt{\hat{q}(\varepsilon k\pi/L)}\xi_k(t) e^{i\pi(k-N_\varepsilon)x/L} \\ &\stackrel{\text{law}}{=} c_L \sum_{k=-N_\varepsilon}^\infty \sqrt{\hat{q}(\pi\varepsilon(N_\varepsilon + k)/L)}\xi_k(t) e^{i\pi kx/L}\end{aligned}$$

$$\approx c_L \sum_{k\in\mathbf{Z}} \sqrt{\hat q(1)}\xi_k(t)e^{i\pi kx/L} \approx \sqrt{\hat q(1)}\,\eta(x,t)\;.$$

In this equation, we justify the passage from the second to the third line by the fact that the linear part of (GL) damps high frequencies, so contributions from Fourier modes beyond $k \approx \varepsilon^{-1}$ can be neglected. Furthermore, $\pi\varepsilon(N_\varepsilon + k)/L \to 1$ for $\varepsilon \to 0$.

Plugging the previous equation into (2.6), we obtain (GL). The aim of the present article is to make this formal calculation rigorous.

Remark 2.4 The approach outlined above relies on the presence of a stable cubic (or higher order) nonlinearity. For the moment, we cannot treat quadratic nonlinearities like the one arising in convection problems. See however [Blö03b] for a result on bounded domains covering that situation or [Sch99] for a deterministic result in unbounded domains.

Remark 2.5 Even though we restrict ourselves to the case of the stochastic Swift-Hohenberg equation, it is clear from the above formal calculation that one expects similar results to hold for a much wider class of equations. In fact, the linear result is proved for a quite general class of operators $P(i\partial_x)$ (cf. Section 7). Furthermore, the main result of this paper, Theorem 1.1, is expected to hold for Stochastic PDE of the type

$$\partial_t U = -P(i\partial_x)\,U + \varepsilon^2\nu U - \mathcal{F}(U) + \varepsilon^{\frac{3}{2}}\xi_\varepsilon\;,$$

with periodic boundary conditions on $D_\varepsilon = [-L\varepsilon^{-1}, L\varepsilon^{-1}]$, for a large class of stable cubic (or higher order) nonlinearities $\mathcal{F}(\cdot)$.

Before we proceed with the proofs of the results stated in the introduction, let us introduce a few more notations that will be useful for the rest of this article.

2.1 Notations, projections, and spaces

We already introduced the L^2-spaces $\mathcal{H}_a$ and $\mathcal{H}_u$, as well as the operators π_ε and ι_ε. We will denote by $e_k(x) = e^{ik\pi x/L}/\sqrt{2L}$ the complex orthonormal Fourier basis in $\mathcal{H}_a$.

Definition 2.6 We define the scale of (fractional) Sobolev spaces $\mathcal{H}_a^\alpha \subset \mathcal{H}_a$ with $\alpha \in \mathbf{R}$ as the closure of the set of $2L$-periodic complex-valued trigonometric polynomials $A = \sum A_k e_k$ under the norm $\|A\|_{a,\alpha}^2 = \sum_k (1+|k|)^{2\alpha}|A_k|^2$. We also define the space $\mathcal{H}_u^\alpha$ as those real-valued functions u such that $\iota_\varepsilon u \in \mathcal{H}_a^\alpha$. We endow these spaces with the natural norm $\|u\|_{u,\alpha} = \|\iota_\varepsilon u\|_{a,\alpha}$.

We also denote by L_a^p (respectively L_u^p) the complex (respectively real) space $\mathrm{L}^p([-L,L])$, endowed with the usual norm. We similarly define the spaces $\mathcal{C}_a^0$ and $\mathcal{C}_u^0$ of periodic continuous bounded functions. We will from time to time consider e_k as elements of $\mathcal{H}_a^\alpha$, L_a^p, or the complexifications of $\mathcal{H}_u^\alpha$ and L_u^p.

Note that with this notation, we have

$$\iota_\varepsilon \pi_\varepsilon e_k = \begin{cases} e_k & \text{if } k \ge -N_\varepsilon, \\ e_{-k-2N_\varepsilon} & \text{if } k < -N_\varepsilon. \end{cases}$$

In particular, one has $\|\pi_\varepsilon e_k\|_{u,\alpha} \le \|e_k\|_{a,\alpha}$ for every $\alpha \ge 0$.

Remark 2.7 Although the norm in $\mathcal{H}_u^\alpha$ is equivalent to the standard α-Sobolev norm, the equivalence constants depend on ε. In particular, the operators $\iota_\varepsilon : \mathcal{H}_u^\alpha \to \mathcal{H}_a^\alpha$ and $\pi_\varepsilon : \mathcal{H}_a^\alpha \to \mathcal{H}_u^\alpha$ are bounded by 1 with our choice of norms, which would not be the case if $\mathcal{H}_u^\alpha$ was equipped with the standard norm instead.

Remark 2.8 Since the injection $\iota_\varepsilon : \mathcal{H}_u^1 \to \mathcal{H}_a^1$, the inclusion $\mathcal{H}_a^1 \hookrightarrow \mathcal{C}_a^0$, as well as the projection $\pi_\varepsilon : \mathcal{C}_a^0 \to \mathcal{C}_u^0$ are all bounded independently of ε, the inclusion $\mathcal{H}_u^1 \hookrightarrow \mathcal{C}_u^0$, which is given by the composition of these three operators, is also bounded independently of ε.

Finally, we define, for some sufficiently small constant $\delta > 0$, the projections $\Pi_{\delta/\varepsilon}$ and $\Pi_{\delta/\varepsilon}^c$by

$$\Pi_{\delta/\varepsilon}\Big(\sum_{k\in\mathbf{Z}} \gamma_k e^{ik\pi x/L}\Big) = \sum_{|k|\le\delta/\varepsilon} \gamma_k e^{ik\pi x/L} \quad \text{and} \quad \Pi_{\delta/\varepsilon}^c = 1 - \Pi_{\delta/\varepsilon}\;. \tag{2.7}$$

3 Bounds on the Residual

Our first step in the proof of Theorem 1.1 is to control the residual (defined in Definition 3.3 below), which measures how well a given approximation satisfies the mild formulation of (SH$_\varepsilon$). Before we give the definition of a mild solution, we define the stochastic convolutions $W_{\mathcal{L}_\varepsilon}(t)$ and $W_{\Delta_\varepsilon}(t)$, which are formally the solutions to the linear equations:

$$W_{\mathcal{L}_\varepsilon}(t) = \sqrt{Q_\varepsilon}\int_0^t e^{(t-\tau)\mathcal{L}_\varepsilon}\, dW_\xi(t) \tag{3.1a}$$

$$W_{\Delta_\varepsilon}(t) = \sqrt{\hat q(1)}\int_0^t e^{(t-\tau)\Delta_\varepsilon}\, dW_\eta(t)\;. \tag{3.1b}$$

Here $W_\xi(t)$ and $W_\eta(t)$ denote standard cylindrical Wiener processes (*i.e.* space-time white noises). Note that W_ξ is real valued, while W_η is complex valued.

The covariance operator Q_ε is given by the convolution with q_ε as mentioned in (1.1). We will assume throughout this section the following.

Assumption 3.1 *The kernel q_ε can be chosen in a way such that there exists a constant C and a joint realisation of $W_{\mathcal{L}_\varepsilon}$ and W_{Δ_ε} such that*

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \|W_{\mathcal{L}_\varepsilon}(t) - \pi_\varepsilon W_{\Delta_\varepsilon}(t)\|_{\mathcal{C}_u^0}^p\Big) \le C\varepsilon^{\frac{p}{2}-\kappa}\;,$$

for every $\varepsilon \in (0,1)$.

Remark 3.2 We will prove in Section 7 below that it is always possible to satisfy Assumption 3.1 provided q satisfies some weak regularity and decay conditions.

With these notations, a mild solution, see *e.g.* [DPZ92, p. 182], of the rescaled equation (SH$_\varepsilon$) is a process u with continuous paths such that:

$$u(t) = e^{t\mathcal{L}_\varepsilon} u(0) + \int_0^t e^{(t-\tau)\mathcal{L}_\varepsilon} (\tilde{\nu} u(\tau) - u^3(\tau)) d\tau + W_{\mathcal{L}_\varepsilon}(t) \;, \tag{3.2}$$

almost surely. We also consider mild solutions A of (GL)

$$A(t) = e^{t\Delta_\varepsilon} A(0) + \int_0^t e^{(t-\tau)\Delta_\varepsilon} (\tilde{\nu} A(\tau) - 3|A(\tau)|^2 A(\tau))\, d\tau + W_{\Delta_\varepsilon}(t) \;. \tag{3.3}$$

This motivates the following definition:

Definition 3.3 Let ψ be an $\mathcal{H}_u$-valued process. The *residual* Res(ψ) of ψ is the process given by

$$\text{Res}(\psi)(t) = -\psi(t) + e^{t\mathcal{L}_\varepsilon} \psi(0) + \int_0^t e^{(t-\tau)\mathcal{L}_\varepsilon} (\tilde{\nu} \psi(\tau) - \psi^3(\tau)) d\tau + W_{\mathcal{L}_\varepsilon}(t) \;, \tag{3.4}$$

where $W_{\mathcal{L}_\varepsilon}(t)$ is as in (3.1a).

It measures how well the process ψ approximates a mild solution of (SH$_\varepsilon$). Let us now introduce the concept of admissible initial condition. Since we are dealing with a family of equations parametrised by $\varepsilon \in (0,1)$, we actually consider a family of initial conditions. We emphasise on the ε-dependence here, but we will always consider it as implicit in the sequel.

Definition 3.4 A family of random variables A^ε with values in $\mathcal{H}_a$ (or equivalently a family μ^ε of probability measures on $\mathcal{H}_a$) is called *admissible* if there exists a decomposition $A^\varepsilon = W_0^\varepsilon + A_1^\varepsilon$, a constant C_0, and a family of constants $\{C_q\}_{q \geq 1}$ such that

1. $A_1^\varepsilon \in \mathcal{H}_a^1$ almost surely and $\mathbb{E}\|A_1^\varepsilon\|_{a,1}^q \leq C_q$ for every $q \geq 1$,
2. the W_0^ε are centred Gaussian random variables such that

$$|\mathbb{E}\langle e_k, W_0^\varepsilon\rangle\langle e_\ell, W_0^\varepsilon\rangle| \leq C_0 \frac{\delta_{k\ell}}{1+|k|^2} \;, \tag{3.5}$$

for all $k, \ell \in \mathbf{Z}$, ($\delta_{k\ell} = 1$ for $k = \ell$ and 0 otherwise)

and such that these bounds are independent of ε. A family of random variables u^ε with values in $\mathcal{H}_u$ is called admissible if $\iota_\varepsilon u^\varepsilon$ is admissible.

Remark 3.5 The definition above is consistent with the definition of π_ε in the sense that if A^ε is admissible, then $\pi_\varepsilon A^\varepsilon$ is also admissible.

Remark 3.6 Note that (3.5) implies that the covariance operator of W_0^ε commutes with the Laplacian, so that $W_0^\varepsilon \stackrel{\text{law}}{=} \sum_{k\in\mathbf{Z}} c_k^\varepsilon \xi_k e_k$, where $c_k^\varepsilon \leq C/(1+|k|)$ and the ξ_k are independent normal random variables with the restriction that $\xi_{-k} = \overline{\xi_k}$. This implies by Lemma A.1 that $\mathbb{E}\|W_0^\varepsilon\|_{\mathcal{C}_a^0}^p \leq C$ for every $p \geq 1$, as $\|e_k\|_{L^\infty} \leq C$ and $\mathrm{Lip}(e_k) \leq C|k|$.

We have the following result.

Theorem 3.7 (Residual) *Let Assumption 3.1 be satisfied. Then, for every $p \geq 1$, $T_0 > 0$, $\kappa > 0$, and admissible initial condition $A(0)$, there is a constant $C_{\kappa,p} > 0$ such that the mild solution A of (GL) with initial condition $A(0)$ satisfies*

$$\mathbb{E}\Big(\sup_{t\in[0,T_0]} \|\mathrm{Res}(\pi_\varepsilon A)(t)\|_{\mathcal{C}_u^0}^p\Big) \leq C_{\kappa,p}\, \varepsilon^{\frac{p}{2}-\kappa}. \tag{3.6}$$

For the proof of the theorem we need two technical lemmas. The first one provides us with estimates on the operator norm for the difference between the semigroup of the original equation and that of the amplitude equation.

Lemma 3.8 *Let H_t be defined as*

$$H_t := e^{-\mathcal{L}_\varepsilon t}\pi_\varepsilon - \pi_\varepsilon e^{-\Delta_\varepsilon t}\;. \tag{3.7}$$

Then for all $\alpha > 0$ there exists a constant $C > 0$ such that

$$\|H_t\|_{\mathcal{L}(\mathcal{H}_a,\mathcal{H}_u^\alpha)} \leq C\varepsilon t^{-\frac{\alpha+1}{2}} \quad \textit{and} \quad \|H_t\|_{\mathcal{L}(\mathcal{H}_a^1,\mathcal{C}_u^0)} \leq C\varepsilon^{1/2}\;. \tag{3.8}$$

Proof. The operator H_t acts on $e_k \in \mathcal{H}_a$ as

$$H_t e_k = \lambda_k(t)\, \pi_\varepsilon e_k\;, \tag{3.9}$$

where the $\lambda_k(t)$'s are given by

$$\lambda_k(t) = c e^{-t\left(1+\varepsilon^{-2}\left(1-\frac{\varepsilon^2\pi^2}{L^2}(k-N_\varepsilon)^2\right)^2\right)} - c e^{-t\left(1+4\left(\frac{k\pi}{L}-\delta_\varepsilon\right)^2\right)}, \tag{3.10}$$

with some constant c bounded by 1. By Taylor expansion around $k=0$, we easily derive for some constants c and C the bound

$$|\lambda_k(t)| \leq \begin{cases} C & \text{for all } k \in \mathbf{Z}, \\ Ct\varepsilon(1+|k|)^3 e^{-ct(1+|k|)^2} & \text{for } |k| \leq N_\varepsilon, \end{cases} \tag{3.11}$$

Let now $h = \sum_{k\in\mathbf{Z}} h_k e_k \in \mathcal{H}_a$. We write

$$\|H_t h\|_{u,\alpha} \leq \|H_t \Pi_{\delta/\varepsilon} h\|_{u,\alpha} + \|H_t \Pi_{\delta/\varepsilon}^c h\|_{u,\alpha}$$

for $\delta > 0$ sufficiently small so that $\delta/\varepsilon \leq N_\varepsilon$. It follows furthermore from standard analytic semigroup theory that H_t is bounded by $Ct^{-(\alpha+1)/2}$ as an operator from

$\mathcal{H}_a^{-1}$ into $\mathcal{H}_u^\alpha$. Since the operator $\Pi_{\delta/\varepsilon}^c : \mathcal{H}_a \to \mathcal{H}_a^{-1}$ is bounded by $C\varepsilon$, it follows that one has indeed $\|H_t\Pi_{\delta/\varepsilon}^c h\|_{u,\alpha} \le C\varepsilon t^{-(\alpha+1)/2}\|h\|_a$. The term $\|H_t\Pi_{\delta/\varepsilon} h\|_{u,\alpha}$ is in turn bounded by

$$\begin{aligned}\|H_t\Pi_{\delta/\varepsilon}h\|_{u,\alpha}^2 &\le Ct^2\varepsilon^2 \sum_{|k|\le\delta/\varepsilon} (1+|k|)^{6+2\alpha} e^{-ct(1+|k|)^2}|h_k|^2 \\ &\le Ct^{-\alpha-1}\varepsilon^2 \sum_{|k|\le\delta/\varepsilon} (t(1+|k|)^2)^{3+\alpha} e^{-ct(1+|k|)^2}|h_k|^2 \\ &\le Ct^{-\alpha-1}\varepsilon^2\|h\|_a^2\;,\end{aligned}$$

from which the first bound follows. To show the second bound, take $h = \sum_k h_k e_k$ in $\mathcal{H}_a^1$. Now a crude estimate shows

$$\|H_t h\|_{\mathcal{C}_u^0} \le C\sum_{k\in\mathbb{Z}} |\lambda_k(t)|\,|h_k| \le C\sqrt{\sum_{k\in\mathbb{Z}} \frac{|\lambda_k(t)|^2}{1+|k|^2}}\|h\|_{a,1}\;. \tag{3.12}$$

It follows from (3.11) that

$$|\lambda_k(t)|^2/(1+|k|^2) \le C\min\{\varepsilon^2, 1/(1+|k|^2)\}\;, \tag{3.13}$$

so that $\sum_{k\in\mathbb{Z}} \frac{|\lambda_k(t)|^2}{1+|k|^2} \le C\varepsilon$ by treating separately the case $|k| \le \varepsilon^{-1}$ and the case $|k| > \varepsilon^{-1}$. □

The second technical lemma bounds the difference between the linear part of the original equation and that of the amplitude equation, applied to an admissible initial condition. The idea is that, for an initial condition which admits the decomposition $A = W_0 + A_1$, one can use the $\mathcal{H}_a^1$-regularity to control the term involving A_1 and Gaussianity to control the term involving W_0.

Lemma 3.9 *Let A be admissible in the sense of Definition 3.4 and let H_t be defined by (3.7). Then for every $T_0 > 0$ $\kappa > 0$ and $p \ge 1$ there exist constants $C > 0$ such that*

$$\mathbb{E}\Big(\sup_{t\in[0,T_0]} \|H_tA\|_{\mathcal{C}_u^0}^p\Big) \le C\varepsilon^{\frac{p}{2}-\kappa}. \tag{3.14}$$

Proof. Since A is admissible, it can be written as $A = W_0 + A_1$ with the same notations as in Definition 3.4. The bound on H_tA_1 is an immediate consequence of Lemma 3.8 above, so we only consider the term involving W_0. We write $W_0 = \sum_{k\in\mathbf{Z}} c_k^\varepsilon \xi_k e_k$ as in Remark 3.6, so that by (3.9)

$$H_tW_0 = \sum_{k\in\mathbf{Z}} c_k^\varepsilon\lambda_k(t)\xi_k\,\pi_\varepsilon e_k\;,$$

with λ_k as in (3.10). We use now Lemma A.1 with domain $G = [-L,L]\times[0,T_0]$ and

$$f_k(x,t) = c_k^\varepsilon\lambda_k(t)\,(\pi_\varepsilon e_k)(x)\;.$$

From (3.13), we derive $\|f_k\|_{L^\infty} \le C\min\{\varepsilon, 1/(1+|k|)\}$. Furthermore, it is easy to see by a crude estimate on Lip(λ_k) that Lip(f_k) $\le C\varepsilon^{-4}(1+|k|)^4$ for some constant C, so that the required bound follows. Note that any bound on Lip(f_k) which is polynomial in ε^{-1} and $|k|$ is sufficient. □

Proof of Theorem 3.7. We start be reformulating the residual in a more convenient way. We add and subtract $\int_0^t e^{(t-\tau)\mathcal{L}_\varepsilon}(\pi_\varepsilon 3A|A|^2)(\tau)\,d\tau$ to obtain

$$\begin{aligned}\mathrm{Res}(\pi_\varepsilon A)(t) &= -(\pi_\varepsilon A)(t) + e^{t\mathcal{L}_\varepsilon}(\pi_\varepsilon A)(0) + W_{\mathcal{L}_\varepsilon}(t)\\ &\quad + \tilde\nu \int_0^t e^{(t-\tau)\mathcal{L}_\varepsilon}\big(\tilde\nu(\pi_\varepsilon A)(\tau) - ((\pi_\varepsilon A)(\tau))^3\big)\,d\tau\\ &= H_t A(0) + \int_0^t H_{t-\tau}\big(\tilde\nu\varepsilon A(\tau) - (A(\tau))^3\big)\,d\tau\\ &\quad + \int_0^t e^{(t-s)\mathcal{L}_\varepsilon}\big((\pi_\varepsilon 3|A|^2 A)(\tau) - ((\pi_\varepsilon A)(\tau))^3\big)\,d\tau\\ &\quad + W_{\mathcal{L}_\varepsilon}(t) - \pi_\varepsilon W_{\Delta_\varepsilon}(t)\,,\end{aligned}$$

where the operator H_t is defined in (3.7). We estimate each term in the above expression separately, starting with the one involving the initial conditions. Since we have assumed that $A(0)$ is admissible, Lemma 3.9 applies and we obtain

$$\mathbb{E}\sup_{t\in[0,T]}\|H_t A(0)\|^p_{\mathcal{C}^0_u} \le C_p \varepsilon^{\frac{p}{2}-\kappa}.$$

Furthermore, Assumption 3.1 ensures that $W_{\mathcal{L}_\varepsilon}(t) - \pi_\varepsilon W_{\Delta_\varepsilon}(t)$ satisfies the requested bound.

We now use Lemma 3.8 for some $\alpha \in (\frac{1}{2}, 1)$ together with the embedding of $\mathcal{H}^\alpha_a$ in $\mathcal{C}^0_a$ to deduce that:

$$\begin{aligned}\Big\|\int_0^t H_{t-\tau}\Big(\tilde\nu\varepsilon A(\tau) - (A(\tau))^3\Big)\,d\tau\Big\|_{\mathcal{C}^0_u} &\le C\int_0^t \|H_{t-\tau}\|_{\mathcal{L}(L^2_a,\mathcal{H}^\alpha_a)}\,d\tau \sup_{0\le\tau\le t}\|A(\tau)\|^3_{L^6_a}\\ &\le C\varepsilon\int_0^t (t-\tau)^{-\frac{\alpha+1}{2}}\,d\tau \sup_{0\le\tau\le t}\|A(\tau)\|^3_{L^6_a}\\ &\le C\varepsilon \sup_{0\le\tau\le t}\|A(\tau)\|^3_{L^6_a}\ .\end{aligned}$$

Thus with the a–priori estimate on the solution of the amplitude equation from Proposition A.5

$$\mathbb{E}\sup_{t\in[0,T]}\Big\|\int_0^t H_{t-\tau}\big((\nu+1)\varepsilon A(\tau) - (A(\tau))^3\big)\,d\tau\Big\|^p_{\mathcal{C}^0_u} \le C_p\varepsilon^p\ .$$

Let us turn to the remaining term. We have (writing $\tilde e_{2N_\varepsilon} = e^{2i\pi N_\varepsilon x/L}$)

$$\int_0^t e^{(t-\tau)\mathcal{L}_\varepsilon}(3\pi_\varepsilon(|A|^2 A)(\tau) - (\pi_\varepsilon A(\tau))^3)\,d\tau = \int_0^t e^{(t-\tau)\mathcal{L}_\varepsilon}\pi_\varepsilon\big(A(\tau)^3\tilde e_{2N_\varepsilon}\big)\,d\tau$$

$$
\begin{aligned}
&= \int_0^t \pi_\varepsilon e^{(t-\tau)\Delta_\varepsilon}\big(A(\tau)^3 \tilde e_{2N_\varepsilon}\big)\,d\tau \\
&\quad + \int_0^t H_{t-\tau}\big(A(\tau)^3 \tilde e_{2N_\varepsilon}\big)\,d\tau. \\
&=: I_1(t) + I_2(t).
\end{aligned}
$$

Let us consider first $I_2(t)$. We use Lemma 3.8, together with the *a priori* estimate on A from Proposition A.5 to obtain:

$$\mathbb{E} \sup_{t\in[0,T]} \|I_2(t)\|^p_{\mathcal{C}^0_u} \le C_p \varepsilon^p.$$

Now we turn to $I_1(t)$. By Theorem A.7, since we have assumed that the initial conditions are admissible, we know that $A(t)$ is concentrated in Fourier space:

$$\mathbb{E} \sup_{t\in[0,T_0]} \|\Pi^c_{\delta/\varepsilon} A(t)\|^p_{\mathcal{C}^0_a} \le C\varepsilon^{\frac{p}{2}-\kappa}.$$

Consequently we have $A^3 = (\Pi_{\delta/\varepsilon}A)^3 + Z$, where

$$\mathbb{E} \sup_{t\in[0,T_0]} \|Z\|^p_{\mathcal{C}^0_a} \le C\varepsilon^{\frac{p}{2}-\kappa} \quad \text{and} \quad \mathbb{E} \sup_{t\in[0,T_0]} \|\Pi_{\delta/\varepsilon} A(t)\|^p_{\mathcal{C}^0_a} \le C. \tag{3.15}$$

Furthermore, we know that $(\Pi_{\delta/\varepsilon}A)^3 e_{2N_\varepsilon}$ has non-vanishing Fourier coefficients only for wavenumbers between $2N_\varepsilon - 3\delta/\varepsilon$ and $2N_\varepsilon - 3\delta/\varepsilon$. By choosing $\delta < 2/3$, say $\delta = 1/3$, we thus guarantee the existence of constants C and c independent of ε such that

$$\|e^{t\Delta_\varepsilon}(\Pi_{\delta/\varepsilon}A)^3 e_{2N_\varepsilon}\|_{\mathcal{C}^0_a} \le C\varepsilon^{-1} e^{-c\varepsilon^{-2}t} \|(\Pi_{\delta/\varepsilon}A)^3\|_{\mathcal{C}^0_a}\ .$$

Hence,

$$
\begin{aligned}
&\Big\|\int_0^t \pi_\varepsilon e^{(t-\tau)\Delta_\varepsilon}\Big(\big(\Pi_{\delta/\varepsilon}A(\tau)\big)^3 e^{\frac{2i\pi N_\varepsilon x}{L}}\Big)d\tau\Big\|_{\mathcal{C}^0_u} \\
&\qquad \le C\int_0^t e^{-c\varepsilon^{-2}(t-\tau)}\varepsilon^{-1}\|\Pi_{\delta/\varepsilon}A(\tau)\|^3_{\mathcal{C}^0_a}\,d\tau \\
&\qquad \le C\varepsilon \sup_{t\in[0,T_0]} \|\Pi_{\delta/\varepsilon}A(t)\|^p_{\mathcal{C}^0_a}\ .
\end{aligned}
\tag{3.16}
$$

Since furthermore $\|\pi_\varepsilon e^{t\Delta_\varepsilon}\|_{\mathcal{L}(\mathcal{C}^0_a,\mathcal{C}^0_u)} \le C$ independently of ε, we obtain:

$$\Big\|\int_0^t \pi_\varepsilon e^{(t-\tau)\Delta_\varepsilon}\Big((\Pi^c_{\delta/\varepsilon}A(\tau))^3 e^{\frac{2i\pi N_\varepsilon x}{L}}\Big)d\tau\Big\|_{\mathcal{C}^0_u} \le C \sup_{t\in[0,T_0]} \|\Pi^c_{\delta/\varepsilon}A(t)\|^p_{\mathcal{C}^0_a}\ . \tag{3.17}$$

Combining (3.16), (3.17), and (3.15), we obtain

$$\mathbb{E} \sup_{t\in[0,T]} \|I_1(t)\|^p_{\mathcal{C}^0_u} \le C_p \varepsilon^{\frac{p}{2}}.$$

Putting all the above estimates together we obtain (3.6) of Theorem 3.7. □

4 Main Approximation Result

This section is devoted to the proof of the following approximation theorem.

Theorem 4.1 (Approximation) *Fix $T_0 > 0$, $p \geq 1$, and $\kappa > 0$. There exist joint realisations of the Wiener processes W_ξ and W_η from (3.1) such that, for every admissible initial condition $A(0)$, there exists $C > 0$ such that*

$$\mathbb{E}\Big(\sup_{t\in[0,T_0]} \|u(t) - \pi_\varepsilon A(t)\|^p_{\mathcal{C}^0_u}\Big) \leq C\varepsilon^{\frac{p}{2}-\kappa} . \tag{4.1}$$

where A is the solution of (3.3) with initial condition $A(0)$ and u is the solution of (3.2) with initial condition $u(0) = \pi_\varepsilon A(0)$.

Before we turn to the proof of this result, we make a few preliminary calculations. Let $A(t)$ and $u(t)$ be as in the statement of Theorem 4.1 and define

$$R(t) = u(t) - \pi_\varepsilon A(t) .$$

From (3.2) and Definition 3.3 we easily derive

$$R(t) = \int_0^t e^{(t-\tau)\mathcal{L}_\varepsilon}[\tilde\nu R(\tau) - 3R(\tau)(\pi_\varepsilon A(\tau))^2 - 3R(\tau)^2\pi_\varepsilon A(\tau) - R(\tau)^3]d\tau \\ + \mathrm{Res}(\pi_\varepsilon A)(t).$$

Define

$$\varphi(t) = \mathrm{Res}(\psi)(t), \quad \psi(t) = \pi_\varepsilon A(t)$$

and

$$r(t) = R(t) - \varphi(t). \tag{4.2}$$

Then $r(t)$ satisfies the equation

$$\partial_t r = \mathcal{L}_\varepsilon r + \tilde\nu(r+\varphi) - 3(r+\varphi)\psi^2 - 3(r+\varphi)^2\psi - (r+\varphi)^3, \quad r(0) = 0. \tag{4.3}$$

With these notations, we have the following *a priori* estimates in L^2.

Lemma 4.2 *Under the assumptions of Theorem 4.1 there exists a constant $C > 0$ such that*

$$\mathbb{E}\Big(\sup_{t\in[0,T_0]} \|r(t)\|^p_u\Big) \leq C\varepsilon^{\frac{p}{2}-\kappa} , \tag{4.4}$$

for $r(t)$ defined in (4.2).

Proof. As before, we are using $\|\cdot\|_u$ to denote the norm in $\mathcal{H}_u$ and we denote by $\langle\cdot,\cdot\rangle_u$ the corresponding scalar product. Taking the scalar product of (4.3) with r we obtain

$$\frac{d}{dt}\|r\|^2_u = 2\langle\mathcal{L}_\varepsilon r, r\rangle_u + 2\tilde\nu\langle r+\varphi, r\rangle_u - 6\langle(r+\varphi)\psi^2, r\rangle_u$$

$$\begin{aligned}&-6\langle (r+\varphi)^2\psi, r\rangle_u - 2\langle (r+\varphi)^3, r\rangle_u\\ &=: I_1+I_2+I_3+I_4+I_5\ .\end{aligned}$$

Since $\mathcal{L}_\varepsilon + 1$ is by definition a non-positive selfadjoint operator, we have $I_1 \leq -2\|r\|_u^2$. Moreover, the Cauchy-Schwarz inequality yields:

$$I_2 \leq C\|r\|_u^2 + C\|\varphi\|_u^2\ .$$

It follows from the Young and Cauchy-Schwarz inequalities that

$$I_3 \leq -3\int_{-L}^{L} r^2\psi^2\,dx + C\|r\|_u^2 + C\|\varphi\|_{\mathcal{C}_u^0}^2\|\psi\|_{\mathcal{C}_u^0}^4\ ,$$

and

$$\begin{aligned}I_4 &= -3\int_{-L}^{L} r^3\psi\,dx - 3\int_{-L}^{L} r^2\varphi\psi\,dx - 3\int_{-L}^{L} r\varphi^2\psi\,dx\\ &\leq \frac{1}{8}\|r\|_{L_u^4}^4 + C\|\psi\|_{\mathcal{C}_u^0}^4 + C\|\varphi\|_{\mathcal{C}_u^0}^2\|\psi\|_u^2\ .\end{aligned}$$

Finally, expanding I_5 yields

$$I_5 \leq -\frac{7}{8}\|r\|_{L_u^4}^4 + C\|\varphi\|_{\mathcal{C}_u^0}^4\ .$$

Putting all these bounds together, we obtain:

$$\partial_t\|r\|_u^2 \leq C\|r\|_u^2 + C\Big(1+\|\psi\|_{\mathcal{C}_u^0}^4\Big)\|\varphi\|_{\mathcal{C}_u^0}^2\Big(1+\|\varphi\|_{\mathcal{C}_u^0}^2\Big)\ .$$

We apply now a comparison argument to deduce ($r(0)=0$ by definition)

$$\|r(t)\|_u^2 \leq C\int_0^t e^{C(t-\tau)}\Big(1+\|\psi\|_{\mathcal{C}_u^0}^4\Big)\|\varphi\|_{\mathcal{C}_u^0}^2\Big(1+\|\varphi\|_{\mathcal{C}_u^0}^2\Big)(\tau)d\tau. \tag{4.5}$$

From Theorem 3.7 we derive with $\varphi(t) = \text{Res}(\pi_\varepsilon A)(t)$

$$\mathbb{E}\sup_{t\in[0,T_0]}\|\varphi(t)\|_{\mathcal{C}_u^0}^p \leq C_p\varepsilon^{\frac{p}{2}-\kappa}\ . \tag{4.6}$$

Furthermore, the *a priori* estimate on $A(t)$, Proposition A.5, together with the properties of π_ε yield for $\psi(t) = \pi_\varepsilon A(t)$

$$\mathbb{E}\sup_{t\in[0,T_0]}\|\psi(t)\|_{\mathcal{C}_u^0}^p \leq C_p\ . \tag{4.7}$$

Combining (4.5) with (4.6) and (4.7) we obtain (4.4) of Lemma 4.2. □

To proceed further we first establish two interpolation inequalities. We start by defining the selfadjoint operator

$$\mathcal{A} = \iota_\varepsilon^*(1-\partial_x^2)\iota_\varepsilon\ . \tag{4.8}$$

By Definition 2.6, the $\mathcal{H}_u^\alpha$-norm is given by $\|r\|_{u,\alpha} = \langle r, \mathcal{A}^\alpha r\rangle$. Furthermore, the following interpolation lemma holds.

Lemma 4.3 *For $p \geq 2$ there is a constant $C > 0$ such that*

$$\|u\|_{L^p_u} \leq C\|u\|_{u,1}^{\frac{1}{2}-\frac{1}{p}} \|u\|_u^{\frac{1}{2}+\frac{1}{p}} \quad \text{and} \quad \|u\|_{L^p_u} \leq C\|u\|_{u,2}^{\frac{1}{4}-\frac{1}{2p}} \|u\|_u^{\frac{3}{4}+\frac{1}{2p}}$$

for every $u \in \mathcal{H}^2_u$.

Proof. The proof of the lemma follows from the standard interpolation inequalities, the definition of $\mathcal{A}$ and the properties of the operators ι_ε, π_ε (cf. (2.3) and (2.4)). □

It is also straightforward to verify that $\mathcal{L}_\varepsilon$ and $\mathcal{A}$ have a joint basis of eigenfunctions consisting of $\sin(\pi kx/L)$ and $\cos(\pi kx/L)$. By comparing the eigenvalues it is easy to verify that

$$\langle -\mathcal{L}_\varepsilon u, u\rangle_u \geq \langle \mathcal{A}u, u\rangle_u \quad \text{and thus} \quad \|u\|_{u,1} \leq \|(-\mathcal{L}_\varepsilon)^{\frac{1}{2}} u\|_u \ . \tag{4.9}$$

Furthermore

$$\langle -\mathcal{L}_\varepsilon u, \mathcal{A}u\rangle_u \geq \|\mathcal{A}u\|_u^2 = \|u\|_{u,2}^2 \ . \tag{4.10}$$

We now turn to the

Proof of Theorem 4.1. We take the scalar product of (4.3) with $\mathcal{A}r$ to obtain

$$\begin{aligned} \frac{1}{2}\partial_t \|r\|_{u,1}^2 &= \langle \mathcal{L}_\varepsilon r, \mathcal{A}r\rangle_u + \tilde{\nu}\langle r+\varphi, \mathcal{A}r\rangle_u - 3\langle (r+\varphi)\psi^2, \mathcal{A}r\rangle_u \\ &\quad - 3\langle (r+\varphi)^2\psi, \mathcal{A}r\rangle_u - \langle (r+\varphi)^3, \mathcal{A}r\rangle_u \\ &=: I_1 + I_2 + I_3 + I_4 + I_5 \ . \end{aligned}$$

We then use (4.10) to get $I_1 \leq -\|r\|_{u,2}^2$. Moreover, using Cauchy-Schwarz and Young, one has the bounds

$$I_2 \leq C\|r\|_u^2 + C\|\varphi\|_u^2 + \frac{1}{8}\|r\|_{u,2}^2$$

and

$$I_3 \leq C\|r\|_u^2 \|\psi\|_{\mathcal{C}^0_u}^4 + C\|\varphi\|_u^2 \|\psi\|_{\mathcal{C}^0_u}^4 + \frac{1}{8}\|r\|_{u,2}^2 \ .$$

In order to bound the term I_4 we use Lemma 4.3 with $p = 4$:

$$I_4 = \frac{1}{8}\|r\|_{u,2}^2 + C\|\psi\|_{\mathcal{C}^0_u}^{\frac{8}{3}} \|r\|_u^{\frac{14}{3}} + C\|\psi\|_{\mathcal{C}^0_u}^2 \|\varphi\|_{\mathcal{C}^0_u}^4 \ .$$

Finally, we use Lemma 4.3 with $p = 6$ to bound I_5:

$$I_5 \leq \delta\|r\|_{u,2}^2 + C_\delta\|\varphi\|_{\mathcal{C}^0_u}^6 + C_\delta\|r\|_u^{10} \ .$$

Putting everything together we obtain:

$$\begin{aligned} \partial_t \|r\|_{u,1}^2 &\leq C\|r\|_u^2 \Big(\|\psi\|_{\mathcal{C}^0_u}^4 + \|\psi\|_{\mathcal{C}^0_u}^3 \|r\|_u^2 + \|\psi\|_{\mathcal{C}^0_u}^2 \|r\|_u^4 + \|r\|_u^8 \Big) \\ &\quad + C\|\varphi\|_{\mathcal{C}^0_u}^2 \Big(1 + \|\varphi\|_{\mathcal{C}^0_u}^2 \|\psi\|_{\mathcal{C}^0_u}^2 + \|\psi\|_{\mathcal{C}^0_u}^4 + \|\varphi\|_{\mathcal{C}^0_u}^4 \Big) \ . \end{aligned} \tag{4.11}$$

Estimate (4.1) follows now from (4.11), together with Lemma 4.2 and the *a priori* bounds on φ and ψ from (4.7) and (4.6). □

5 Attractivity

This section provides attractivity results for the SPDE. We consider the rescaled equation (SH_ε), and we prove that regardless of the initial condition $u(0)$ we start with, we will end up for sufficiently large $t > 0$ with an admissible $u(t)$, thus giving admissible initial conditions for the amplitude equation. The main result of this section is contained in the following theorem.

Theorem 5.1 (Attractivity) *For all (random) initial conditions $u(0)$ such that $u(0) \in \mathcal{H}_u$ almost surely and every $t > 0$, the mild solution $u(t)$ of (SH_ε) is admissible in the sense of Definition 3.4. Furthermore, given a $T_0 > 0$ the family of constants $\{C_q\}_{q>0}$ which appears in the definition of admissibility is independent of the initial conditions and the time t for $t > T_0$.*

Remark 5.2 In [Cer99] and [GM01] uniform bounds on the solutions after transient times were obtained that are independent of the initial condition. However, the statements given in these papers do not cover the situation presented here.

The rest of this section is devoted to the proof of this theorem. First we will prove standard a-priori estimates in L^2-spaces that rely on the strong nonlinear stability of the equation. Then we will provide regularisation results using the $\mathcal{H}^1_u$ norm which allow us to get to the $\mathcal{C}^0_u$ space and we end with the admissibility of the solution. Note that the solution u will never be in $\mathcal{H}^1$, therefore we have to consider suitable transformations.

Let $u(t)$ denote the mild solution of (SH_ε), *i.e.* a solution of (3.2). Denote as in (3.1a) by $W_{\mathcal{L}_\varepsilon}$ the stochastic convolution for the operator $\mathcal{L}_\varepsilon$ and define $v := u - W_{\mathcal{L}_\varepsilon}$. Then v satisfies the equation

$$\partial_t v = \mathcal{L}_\varepsilon v + \tilde{\nu}(v + W_{\mathcal{L}_\varepsilon}) - (v + W_{\mathcal{L}_\varepsilon})^3, \tag{5.1}$$

with the same initial conditions as u. We start by obtaining an L^2 estimate on u. Before we do this let us discuss some estimates for the stochastic convolution. Using first Proposition 7.1 we obtain

$$\mathbb{E} \sup_{t\in[0,T_0]} \|W_{\mathcal{L}_\varepsilon}(t)\|^{2p}_{\mathcal{C}^0_u} \leq C\mathbb{E} \sup_{t\in[0,T_0]} \|W_{\Delta_\varepsilon}(t)\|^{2p}_{\mathcal{C}^0_a} + C\varepsilon^{p/2-\kappa} .$$

Hence, using the modification of Lemma A.3 or Proposition A.5 with $c = 0$,

$$\mathbb{E} \sup_{t\in[0,T_0]} \|W_{\mathcal{L}_\varepsilon}(t)\|^{2p}_{\mathcal{C}^0_u} \leq C . \tag{5.2}$$

Lemma 5.3 *Let $u(t)$ be the solution of (3.2). Fix arbitrary $T_0 > 0$. Then there exists a constant $C > 0$ independent of $u(0)$ such that*

$$\sup_{t\geq T_0} \mathbb{E}\|u(t)\|^p_u \leq C.$$

Assume further that $\mathbb{E}\|u(0)\|_u^p \le c_0$. Then, given $T_0 > 0$ there exists a constant C such that

$$\sup_{t\ge 0} \mathbb{E}\|u(t)\|_u^p \le C, \quad \text{and} \quad \mathbb{E} \sup_{t\in[0,T_0]} \|u(t)\|_u^p \le C.$$

Proof. We multiply (5.1) with v, integrate over $[-L, L]$, use the dissipativity of $\mathcal{L}_\varepsilon$ in $\mathcal{H}_u$, together with the fact that

$$-\langle v, (v + W_{\mathcal{L}_\varepsilon})^3\rangle_u \le -(1-\delta)\|v\|_u^4 + \delta\|v\|_u^2 + C_\delta\|W_{\mathcal{L}_\varepsilon}\|_u^4$$

for every $\delta > 0$, which we choose to be sufficiently small, to obtain

$$\partial_t\|v\|_u^2 \le -C_1\|v\|_u^4 + C_2\Big(1 + \|W_{\mathcal{L}_\varepsilon}\|_{\mathcal{C}_u^0}^4\Big),$$

for some positive constants C_1 and C_2. A comparison theorem for ODE yields for $t \in [0, T_0]$

$$\begin{aligned}\|v(t)\|_u^2 &\le \max\Big\{C(1 + \sup_{t\in[0,T_0]} \|W_{\mathcal{L}_\varepsilon}\|_{\mathcal{C}_u^0}^2); \frac{1}{C_1 t/2 + 1/\|v(0)\|_u^2}\Big\} \\ &\le C\Big(1 + \sup_{t\in[0,T_2]} \|W_{\mathcal{L}_\varepsilon}\|_{\mathcal{C}_u^0}^2 + \frac{1}{t}\Big) . \end{aligned} \tag{5.3}$$

Note furthermore, that

$$\partial_t\|v\|_u^2 \le -c\|v\|_u^2 + C\Big(1 + \|W_{\mathcal{L}_\varepsilon}\|_{\mathcal{C}_u^0}^4\Big) .$$

Again a comparison argument for ODEs yields for any $T_0 > 0$

$$\|v(t)\|_u^2 \le e^{c(t-T_0)}\|v(T_0)\|_u^2 + C\int_{T_0}^t e^{-c(t-s)}\Big(1 + \|W_{\mathcal{L}_\varepsilon}(s)\|_{\mathcal{C}_u^0}^4\Big)\, ds \tag{5.4}$$

The claims of the lemma follow now easily from (5.3) and (5.4), the fact that $u = v + W_{\mathcal{L}_\varepsilon}$, and the estimates on the stochastic convolution from (5.2). □

Lemma 5.4 *Fix $\delta > 0$, $p > 0$, and $T_0 > 0$. Then there is a constant C such that for all mild the solutions u of (SH$_\varepsilon$) (i.e. (3.2)) with $\mathbb{E}\|u(0)\|_u^{5p} \le \delta$ the following estimate holds*

$$\sup_{t\ge T_0} \mathbb{E}\|u(t)\|_{\mathcal{C}_u^0}^p \le C . \tag{5.5}$$

Proof. Define

$$w(t) := u(t) - e^{t\mathcal{L}_\varepsilon}u(0) - W_{\mathcal{L}_\varepsilon} =: u(t) - \varphi(t)$$

Now w fulfils

$$\partial_t w = \mathcal{L}_\varepsilon w + \tilde\nu(w + \varphi) - (w + \varphi)^3, \quad w(0) = 0 \tag{5.6}$$

Consider $\mathcal{A}$ defined in (4.8) and multiply (5.6) with $\mathcal{A}w$, integrate over $[-L,L]$, use Lemma 4.3 with $p=6$ as well as $\|v\|_{u,1} \le \|v\|_{u,2}$ to obtain:

$$\partial_t \|w\|_{u,1}^2 \le -C_1 \|w\|_{u,1}^2 + C_2\Big(\|w\|_u^2 + \|w\|_u^{10} + \|\varphi\|_u^2 + \|\varphi\|_{L_u^6}^6\Big)$$

A comparison theorem for ODE now yields:

$$\|w(t)\|_{u,1}^2 \le C_2 \int_0^t e^{-C_1(t-\tau)} (1 + \|w\|_u^{10} + \|\varphi\|_{L_u^6}^6)(\tau)\, d\tau \;. \tag{5.7}$$

Using (4.9) and Lemma 4.3 we deduce that $\|u\|_{L_u^6} \le C\|(-\mathcal{L}_\varepsilon)^{1/2} u\|_u^{1/3} \|u\|_u^{2/3}$. Hence,

$$\|e^{t\mathcal{L}_\varepsilon} u_0\|_{L_u^6}^3 \le C t^{-1/2} \|u_0\|_u^3 \;. \tag{5.8}$$

Taking the $\mathcal{L}^{p/2}$-norm in probability space, we deduce from (5.7) using (5.8) and the embedding of $\mathcal{H}_u^1$ into $\mathcal{C}_u^0$ from Remark 2.8

$$\begin{aligned}\Big(\mathbb{E}\|w(t)\|_{\mathcal{C}_u^0}^p\Big)^{2/p} &\le C\Bigg(1 + \sup_{t\ge 0} (\mathbb{E}\|w(t)\|_{\mathcal{C}_u^0}^{5p})^{2/p} + \sup_{t\ge 0} (\mathbb{E}\|W_{\mathcal{L}_\varepsilon}\|_{L_u^6}^{3p})^{2/p}\Bigg) \\ &\quad + C\int_0^t \tau^{-1/2} e^{-C_1\tau} d\tau\, (\mathbb{E}\|u(0)\|_u^{3p})^{2/p} \le C \end{aligned} \tag{5.9}$$

for all $t > 0$, where we used the L^2-bounds from Lemma 5.3. Note that this is the reason, why we need the $5p$-th moment of the initial condition $u(0)$. On the other hand, the bound on the stochastic convolution together with standard properties of analytic semigroups enable us to bound $\varphi(t)$, for t sufficiently large:

$$\|\varphi(t)\|_{\mathcal{C}_u^0} \le C\|e^{t\mathcal{L}_\varepsilon} u(0)\|_{u,1} + \|W_{\mathcal{L}_\varepsilon}\|_{\mathcal{C}_u^0} \le C t^{-1/2} \|u(0)\|_u + \|W_{\mathcal{L}_\varepsilon}\|_{\mathcal{C}_u^0}.$$

Estimate (5.5) now follows from the above estimate, Lemma 5.3, the definition of w and estimate (5.9). □

Proof of Theorem 5.1. First, Lemma 5.3 together with Lemma 5.4 establishes the existence of a time $T_0 > 0$ such that $\mathbb{E}\|u(t)\|_{\mathcal{C}_u^0}^p \le C$ for all $t \ge T_0$. Furthermore, combining (5.7) and (5.9) we immediately get that

$$\mathbb{E}\|w(t)\|_{u,1}^p \le C.$$

Thus, under the assumptions of the previous lemma and using the properties of the stochastic convolution $W_{\mathcal{L}_\varepsilon}(t)$ we conclude that for every $t > 0$ $u(t)$ can be decomposed as

$$u(t) = w(t) + Z(t) + e^{t\mathcal{L}_\varepsilon} u(0) \;,$$

where $w(t) \in \mathcal{H}_u^1$ and $Z(t)$ is a centred Gaussian process in $\mathcal{C}_u^0$. Moreover, $e^{t\mathcal{L}_\varepsilon} u(0)$ is in $\mathcal{H}_u^1$ for any $t > 0$, too. We use now the decomposition

$$u(T_0 + \tau) = \tilde{w}(\tau) + \tilde{Z}(\tau) + e^{\tau\mathcal{L}_\varepsilon} u(T_0) \;,$$

where we consider $u(t)$ as the solution starting at sufficiently large $T_0 > 0$ with initial conditions $u(T_0)$. For $\tau > 0$ sufficiently large the process $\iota_\varepsilon \tilde{Z}(\tau) := \iota_\varepsilon W_{\mathcal{L}_\varepsilon}(\tau)$ (in law) is clearly as in 2. of Definition 3.4. For 1. define $W_0(\tau) := \tilde{w}(\tau) + e^{\tau\mathcal{L}_\varepsilon} u(T_0)$. We obtain from Lemma 5.4 and the analog of (5.9) for $\tilde{w}$ that

$$\mathbb{E}\|W_0(\tau)\|_{u,1}^p \le C_p + C\tau^{-p/2}\mathbb{E}\|u(T_0)\|_u^p \le C \;.$$

Hence, the decomposition $u(t) = W_0(t - T_0) + \tilde{Z}(t - T_0)$ shows the admissibility of $u(t)$, where the constants are independent of $t \ge 2T_0$. □

6 Approximation of the Invariant Measure

First, we denote by $\mathcal{P}_t^\varepsilon$ the semigroup (acting on finite Borel measures) associated to (SH$_\varepsilon$) and by $\mathcal{Q}_t^\varepsilon$ the semigroup associated to (GL). Note that $\mathcal{Q}_t^\varepsilon$ depends on ε, but it is for instance independent of ε for $L \in \varepsilon\pi\mathbf{N}$.

Recall also that the Wasserstein distance $\|\cdot\|_W$ between two measures on some metric space $\mathcal{M}$ with metric d is given by

$$\|\mu_1 - \mu_2\|_W = \inf_{\mu\in\mathcal{C}(\mu_1,\mu_2)} \int_{\mathcal{M}^2} \min\{1, d(f,g)\}\, \mu(df, dg) \;.$$

where $\mathcal{C}(\mu_1, \mu_2)$ denotes the set of all measures on $\mathcal{M}^2$ with j-th marginal μ_j. See for example [Rac91] for detailed properties of this distance.

In the sequel, we will use the notation $\|\mu_1-\mu_2\|_{W,p}$ for the Wasserstein distance corresponding to the L^p-norm $d(f,g) = \|f - g\|_{\mathrm{L}^p}$ for $p \in [1,\infty]$. The main result on the invariant measures is

Theorem 6.1 *Let $\mu_{\star,\varepsilon}$ be an invariant measure for (SH$_\varepsilon$) and let $\nu_{\star,\varepsilon}$ be the (unique) invariant measure for (GL). Then, for every $\kappa > 0$, there exists $C > 0$ such that one has*

$$\|\mu_{\star,\varepsilon} - \pi_\varepsilon^* \nu_{\star,\varepsilon}\|_{W,\infty} \le C\varepsilon^{1/2-\kappa}$$

for every $\varepsilon \in (0,1]$.

Note that $\nu_{\star,\varepsilon}$ is actually independent of ε provided $L \in \varepsilon\pi\mathbf{N}$. As usual, the measure $\pi_\varepsilon^*\nu$ denotes the distribution of π_ε under the measure ν.

Proof. Fix $\kappa > 0$ for the whole proof. From the triangle inequality and the definition of an invariant measure, we obtain

$$\begin{aligned}\|\mu_{\star,\varepsilon} - \pi_\varepsilon^*\nu_{\star,\varepsilon}\|_{W,\infty} &\le \|\mathcal{P}_t^\varepsilon \mu_{\star,\varepsilon} - \pi_\varepsilon^* \mathcal{Q}_t^\varepsilon \iota_\varepsilon^* \mu_{\star,\varepsilon}\|_{W,\infty} \\ &\quad + \|\pi_\varepsilon^* \mathcal{Q}_t^\varepsilon \nu_{\star,\varepsilon} - \pi_\varepsilon^* \mathcal{Q}_t^\varepsilon \iota_\varepsilon^* \mu_{\star,\varepsilon}\|_{W,\infty} \;.\end{aligned} \tag{6.1}$$

Concerning the first term, it follows from Theorem 4.1 that the family of measures $\mu_{\star,\varepsilon}$ is admissible and that

$$\|\mathcal{P}_t^\varepsilon \mu_{\star,\varepsilon} - \pi_\varepsilon^* \mathcal{Q}_t^\varepsilon \iota_\varepsilon^* \mu_{\star,\varepsilon}\|_{W,\infty} \le C\varepsilon^{1/2-\kappa} \;.$$

In order to bound the second term in (6.1), we use the exponential convergence of $\mathcal{Q}_t^\varepsilon \mu$ towards a unique invariant measure. This is a well-known result for SPDEs driven by space-time white noise (cf. *e.g.* Theorem 2.4 of [GM01]), but we need the explicit dependence of the constants on the initial measures. The precise bound required for our proof is given in Lemma 6.2 below.

By Lemma 6.2, there exists $t > 0$ such that

$$\|\mathcal{Q}_t^\varepsilon \mu_{\star,0} - \mathcal{Q}_t^\varepsilon \iota_\varepsilon^* \mu_{\star,\varepsilon}\|_{W,\infty} \leq \frac{1}{2\sqrt{L}} \|\iota_\varepsilon^* \mu_{\star,\varepsilon} - \nu_{\star,\varepsilon}\|_{W,2} ,$$

so that the boundedness in L^∞ of π_ε implies

$$\|\mu_{\star,\varepsilon} - \pi_\varepsilon^* \nu_{\star,\varepsilon}\|_{W,\infty} \leq \frac{1}{2\sqrt{L}} \|\iota_\varepsilon^* \mu_{\star,\varepsilon} - \nu_{\star,\varepsilon}\|_{W,2} + C\varepsilon^{1/2-\kappa} .$$

Since the L^2-norm is bounded by $\sqrt{L}$ times the L^∞-norm, this in turn is smaller than

$$\frac{1}{2}\|\mu_{\star,\varepsilon} - \pi_\varepsilon^* \nu_{\star,\varepsilon}\|_{W,\infty} + \frac{1}{2\sqrt{L}} \|\iota_\varepsilon^* \pi_\varepsilon^* \nu_{\star,\varepsilon} - \nu_{\star,\varepsilon}\|_{W,2} + C\varepsilon^{1/2-\kappa} .$$

It follows from standard energy-type estimates that

$$\mathbb{E} \int_{\mathcal{H}_a^\alpha} \|A\|_\alpha \, \nu_{\star,\varepsilon}(dA) < C_\alpha$$

for every $\alpha < 1/2$, where the constants C_α can be chosen independently of ε. This estimate is a straightforward extension of the results presented in Section A.2.

One therefore has $\|\iota_\varepsilon^* \pi_\varepsilon^* \nu_{\star,\varepsilon} - \nu_{\star,\varepsilon}\|_{W,2} \leq C_\kappa \varepsilon^{1/2-\kappa}$. Plugging these bounds back into (6.1) shows that

$$\|\mu_{\star,\varepsilon} - \pi_\varepsilon^* \nu_{\star,\varepsilon}\|_{W,\infty} \leq \frac{1}{2}\|\mu_{\star,\varepsilon} - \pi_\varepsilon^* \nu_{\star,\varepsilon}\|_{W,\infty} + C_\kappa \varepsilon^{1/2-\kappa} ,$$

and therefore concludes the proof of Theorem 6.1. □

Besides the approximation result, the main ingredient for the above reasoning is:

Lemma 6.2 *For every $\delta > 0$, there exists a time $T = T(\delta)$ independent of ε such that*

$$\|\mathcal{Q}_T^\varepsilon \mu - \mathcal{Q}_T^\varepsilon \nu\|_{W,\infty} \leq \delta \|\mu - \nu\|_{W,2} .$$

Proof. It follows from the Bismut-Elworthy-Li formula combined with standard *a priori* bounds on $\mathcal{Q}_t^\varepsilon$ [EL94, DPZ96, Cer99] that

$$\|\mathcal{Q}_t^\varepsilon \mu - \mathcal{Q}_t^\varepsilon \nu\|_{TV} \leq C(1 + t^{-1/2}) \|\mu - \nu\|_{W,2} ,$$

with a constant C independent of ε.

On the other hand, [GM01] there exist constants C and γ such that

$$\|\mathcal{Q}_t^\varepsilon \mu - \mathcal{Q}_t^\varepsilon \nu\|_{TV} \le Ce^{-\gamma t}\|\mu - \nu\|_{TV} \;. \tag{6.2}$$

These constants may in principle depend on ε. By retracing the constructive argument of Theorem 5.5 in [Hai02] with the binding function

$$G(x,y) = -C(y-x)(1 + \|y-x\|_u^{-1/2}) \;,$$

one can however easily show that the constants in (6.2) can be chosen independently of ε. □

7 Approximation of the Stochastic Convolution

In this section, we give L^∞ bounds in time and in space on the difference between the stochastic convolutions of the original equation and of the amplitude equation. The main result of this section is

Theorem 7.1 *Let $W_{\mathcal{L}_\varepsilon}$ and W_{Δ_ε} be defined as in (3.1), and let the correlation functions q_ε with Fourier coefficients q_k^ε satisfy Assumptions 7.3 and 7.4 below. For every $T > 0$, $\kappa > 0$, and $p \ge 1$ there exists a constant C and a joint realisation of $W_{\mathcal{L}_\varepsilon}$ and W_{Δ_ε} such that*

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \|W_{\mathcal{L}_\varepsilon}(t) - \pi_\varepsilon W_{\Delta_\varepsilon}(t)\|_{\mathcal{C}_u^0}^p\Big) \le C\varepsilon^{\frac{p}{2}-\kappa} \;,$$

for every $\varepsilon \in (0,1)$.

We will actually prove a more general result, see Proposition 7.8 below, which has Theorem 7.1 as an immediate corollary. The general result allows the linear operator $\mathcal{L}_\varepsilon$ to be essentially an arbitrary real differential operator instead of restricting it to the operator $-1-\varepsilon^{-2}(1+\varepsilon^2\partial_x^2)^2$. Our main technical tool is a series expansion of the stochastic convolution together with Lemma A.1, which will be proved in Section A.1 below. The expansion with respect to space is performed using Fourier series. For the expansion in time we do not use Karhunen-Loeve expansion directly, since we do not necessarily need an orthonormal basis to apply Lemma A.1. Our choice of an appropriate basis will simplify the coefficients in the series expansion significantly (cf. Lemma A.2). We start by introducing the assumptions required for the differential operator $P(i\partial_x)$.

Assumption 7.2 *Let P denote an even function $P : \mathbf{R} \to \mathbf{R}$ satisfying the following properties:*

- ***P1*** *P is three times continuously differentiable.*
- ***P2*** *$P(\zeta) \ge 0$ for all $\zeta \in \mathbf{R}$ and $P(0) > 0$.*
- ***P3*** *The set $\{\zeta \mid P(\zeta) = 0\}$ is finite and will be denoted by $\{\pm\zeta_1, \ldots, \pm\zeta_m\}$. Note that $\xi_j \neq 0$.*

P4 $P''(\zeta_j) > 0$ *for* $j = 1, \ldots, m$.

P5 *There exists* $R > 0$ *such that* $P(\zeta) \geq |\zeta|^2$ *for all* ζ *with* $|\zeta| \geq R$.

Note that choosing P even ensures that $P(i\partial_x)$ is a real operator, but our results also hold for non-even P, up to trivial notational complications.

We now make precise the assumptions on the noise that drives our equation. Consider an even real-valued distribution q such that its Fourier transform satisfies $\hat{q} \geq 0$. Then, $q(x)\delta(t)$ is the correlation function for a real distribution-valued Gaussian process $\xi(x,t)$ with $x, t \in \mathbf{R}^2$, *i.e.* a process such that $\mathbb{E}\xi(s,x)\xi(t,y) = \delta(t-s)q(x-y)$. We restrict ourself to correlation functions in the following class:

Assumption 7.3 *The distribution* q *is such that* $\hat{q} \in L^\infty(\mathbf{R})$ *and* $\hat{q}$ *is globally Lipschitz continuous.*

At this point, a small technical difficulty arises from the fact that we want to replace ξ by a $2L/\varepsilon$-periodic translation invariant noise process ξ_ε which is close to ξ in the bulk of this interval. Denote by q^ε the $2L/\varepsilon$-periodic correlation function of ξ_ε and by q_k^ε its Fourier coefficients, *i.e.*

$$q_k^\varepsilon = \int_{-L/\varepsilon}^{L/\varepsilon} q^\varepsilon(x)\, e^{-i\frac{k\pi\varepsilon}{L}x}\, dx\ . \tag{7.1}$$

One natural choice is to take for q^ε the periodic continuation of the restriction of q to $[-L/\varepsilon, L/\varepsilon]$. This does however not guarantee that q^ε is again positive definite. Another natural choice is to define q^ε via its Fourier coefficients by

$$q_k^\varepsilon = \int_{-\infty}^{\infty} q(x)\, e^{-i\frac{k\pi\varepsilon}{L}x}\, dx\ , \tag{7.2}$$

which corresponds to taking $q^\varepsilon(x) = \sum_{n\in\mathbf{Z}} q(x + 2nL/\varepsilon)$. This guarantees that q^ε is automatically positive definite, but it requires some summability of q. Note that for noise with bounded correlation length (*i.e.* support of q uniformly bounded) (7.1) and (7.2) coincide for $\varepsilon > 0$ sufficiently small.

We choose not to restrict ourselves to one or the other choice, but to impose only a rate of convergence of the coefficients q_k^ε towards $\hat{q}(k\pi\varepsilon/L)$:

Assumption 7.4 *Let* q *be as in Assumption 7.3. Suppose there is a non-negative approximating sequence* q_k^ε *that satisfies*

$$\sup_{k\in\mathbf{N}_0} |\sqrt{q_k^\varepsilon} - \sqrt{\hat{q}(k\pi\varepsilon/L)}| \leq C\varepsilon\ ,$$

for all sufficiently small $\varepsilon > 0$.

Example 7.5 *A simple example of noise fulfilling Assumptions 7.3 and 7.4 is given by space-time white noise. Here* $\hat{q}(k) = 1$ *and the natural approximating sequence is* $q_k^\varepsilon = 1$ *for all* k.

A more general class of examples is given by the following lemma.

Lemma 7.6 *Let q be positive definite and such that $x \mapsto (1+|x|^2)\, q(x)$ is in* L^1. *Define q_k^ε either by (7.2) or by (7.1) (in the latter case, we assume additionally that the resulting q^ε are positive definite). Then Assumptions 7.3 and 7.4 are satisfied.*

Proof. This follows from elementary properties of Fourier transforms. □

Let us now turn to the stochastic convolution, which is the solution to the linear equation

$$dW_{\mathcal{L}_\varepsilon}(x,t) = \mathcal{L}_\varepsilon W_{\mathcal{L}_\varepsilon}(x,t)\, dt + \sqrt{Q_\varepsilon}\, dW(x,t)\;, \tag{7.3}$$

where

$$\mathcal{L}_\varepsilon = -1 - \varepsilon^{-2} P(\varepsilon i \partial_x)\;,$$

W is a standard cylindrical Wiener process on $\mathrm{L}^2([-L,L])$, and the covariance operator Q_ε is given by the following definition.

Definition 7.7 Let Assumption 7.4 be true. Define q^ε as the function such that q_k^ε are its Fourier coefficients (cf. (7.1)). Then define Q_ε as the rescaled convolution with q^ε, *i.e.*

$$(Q_\varepsilon f)(x) = \frac{1}{\varepsilon} \int_{-L}^{L} f(y)\, q^\varepsilon\Big(\frac{x-y}{\varepsilon}\Big)\, dy\;.$$

Let us expand $W_{\mathcal{L}_\varepsilon}$ into a complex Fourier series. Denote as usual by $e_k(x) = e^{ik\pi x/L}/\sqrt{2L}$ the complex orthonormal Fourier basis on $[-L,L]$. Define furthermore P^ε by

$$P^\varepsilon(k) = \frac{1}{\varepsilon^2} P\Big(\frac{k\varepsilon\pi}{L}\Big) + 1$$

Since Q_ε commutes with $\mathcal{L}_\varepsilon$, we can write the stochastic convolution as

$$\begin{aligned} W_{\mathcal{L}_\varepsilon}(x,t) &= \sqrt{Q_\varepsilon} \int_0^t e^{\mathcal{L}_\varepsilon(t-s)} dW(x,s) \\ &= \sum_{k=-\infty}^{\infty} \sqrt{q_k^\varepsilon}\, e_k(x) \int_0^t \exp(-P^\varepsilon(k)\,(t-s))\, dw_k(s)\;, \end{aligned}$$

where the $\{w_k\}_{k\in\mathbf{Z}}$ are complex standard Wiener processes that are independent, except for the relation $w_{-k} = \overline{w_k}$. We approximate $W_{\mathcal{L}_\varepsilon}(x,t)$ by expanding P in a Taylor series up to order two around its zeroes. We thus define the approximating polynomials P_j^ε by

$$P_j^\varepsilon(k) = \frac{P''(\zeta_j)\pi^2}{2L^2}\Big(k - \frac{L\zeta_j}{\varepsilon\pi}\Big)^2 + 1\;.$$

With this notation, the approximation $\Phi(x,t)$ is defined by

$$\Phi(x,t) = 2\mathrm{Re} \sum_{j=1}^{m} \sqrt{\hat q(\zeta_j)} \sum_{k=-\infty}^{\infty} e_k(x) \int_0^t \exp(-P_j^\varepsilon(k)(t-s))\, d\tilde w_{k,j}(s)\;, \tag{7.4}$$

where the $\tilde{w}_{k,j}$'s are complex i.i.d. complex standard Wiener processes. At this point, let us discuss a rewriting of Φ which makes the link with the notations used in the rest of this article. We decompose $\frac{L\zeta_j}{\varepsilon\pi}$ into an integer part and a fractional part, so we write it as

$$\frac{L\zeta_j}{\varepsilon\pi} = \delta_j + k_j \;, \qquad \delta_j \in [-\tfrac{1}{2}, \tfrac{1}{2}] \;, \qquad k_j = \Big[\frac{L\zeta_j}{\varepsilon\pi}\Big] \in \mathbf{Z}.$$

As before $[z]$ denotes the nearest integer to $z \geq 0$, with the convention that $[\frac{1}{2}] = 1$. For $z < 0$, we define $[z] = -[-z]$. Extend for $m > 1$ the definition of the Hilbert space $\mathcal{H}_a = \mathrm{L}^2([-L, L], \mathbf{C}^m)$ and the definition of the projection

$$\begin{aligned}\pi_\varepsilon : \mathcal{H}_a &\mapsto \mathcal{H}_u \\ A &\to 2\mathrm{Re} \sum_{j=1}^m A_j(x) e^{\frac{i\pi k_j}{L} x} \;.\end{aligned}$$

With this notation, we can write Φ as $\Phi(t) = \pi_\varepsilon \Phi^a(t)$, where the j-th component of Φ^a solves the equation

$$d\Phi_j^a(t) = \Delta_j \Phi_j^a(t)\, dt + \sqrt{\hat{q}(\zeta_j)}\, \eta_j(t) \;. \tag{7.5}$$

Here, the η_j's are independent complex-valued space-time white noises and the Laplacian-type operator Δ_j is given by

$$\Delta_j = -\frac{P''(\zeta_j)}{2}\Big(i\partial_x + \frac{\pi\delta_j}{L}\Big)^2 \;.$$

Now we can prove the following approximation result.

Proposition 7.8 *Let Assumptions 7.2, 7.3 and 7.4 hold and consider Φ and $W_{\mathcal{L}_\varepsilon}$ as defined in (7.3) and (7.5). Then for every $T > 0$, $\kappa > 0$ and every $p \geq 1$, there exists a constant C and joint realisations of the noises W and η_i such that*

$$\mathbb{E}\Big(\sup_{x\in[-L,L]} \sup_{t\in[0,T]} |\Phi(x,t) - W_{\mathcal{L}_\varepsilon}(x,t)|^p\Big) \leq C\varepsilon^{p/2-\kappa} \;.$$

Remark 7.9 This result can not be generalised to dimensions higher than one, since the stochastic convolution of the Laplace operator with space-time white noise is then not even in L^2. It the zeros of P are degenerate, *i.e.* P behaves like $(k - \zeta_j)^{2d}$ for some $d \in \{2, 3, \ldots\}$ then we would obtain an amplitude equation with higher order differential operator, and we can proceed to higher dimension. The other option would be to use fractional noise in space, which is more regular that space-time white noise. Using the scaling invariance of fractional noise, we would obtain fractional noise in the amplitude equation.

Proof. It will be convenient for the remainder of the proof to distinguish between the positive roots ζ_j and the negative roots $-\zeta_j$ of P, so we define $\zeta_{-j} = -\zeta_j$. We start by writing $\Phi = \sum_{j=1}^m (\Phi^{(j)} + \Phi^{(-j)})$ with

$$\Phi^{(j)}(x) = \begin{cases} \Phi_j^a(x)\, e^{\frac{i\pi k_j}{L}x} & \text{for } j > 0, \\ \overline{\Phi_j^a(x)}\, e^{-\frac{i\pi k_j}{L}x} & \text{for } j < 0. \end{cases} \tag{7.6}$$

For $r > 0$ sufficiently small and R as in **P5**, we decompose **Z** into several regions:

$$K_1^{(j)} = \Big\{k \in \mathbf{Z} \,\Big|\, \Big|\frac{k\varepsilon\pi}{L} - \zeta_j\Big| < r\Big\}\;, \quad K_1 = K_1^{(0)} = \bigcup_{j=1}^m (K_1^{(j)} \cup K_1^{(-j)})\;,$$
$$K_2 = \Big\{k \in \mathbf{Z} \,\Big|\, \Big|\frac{k\varepsilon\pi}{L}\Big| < R\Big\}\;, \qquad K_3 = \mathbf{Z} \setminus K_2\;.$$

We suppose that $r > 0$ is sufficiently small such that the $\{K_1^{(j)}\}_{j=\pm1,\ldots,\pm m}$ are disjoint and such that $0 \notin K_1$. The splitting into K^2 and K^3 is mainly for technical reasons. We denote by $\Pi_1^{(j)}$, Π_2, etc. the corresponding orthogonal projection operators in $L^2([-L, L])$. We also define

$$\gamma_k = \gamma_k^{(0)} = \frac{1}{\varepsilon^2} P\Big(\frac{k\varepsilon\pi}{L}\Big) + 1\;,$$
$$\gamma_k^{(j)} = \frac{P''(\zeta_j)\pi^2}{2L^2}\Big(k - \frac{L\zeta_j}{\varepsilon\pi}\Big)^2 + 1 \quad \text{for } j = \pm1, \ldots, \pm m$$

It is a straightforward calculation, using Taylor expansion and Assumption 7.2, that there exist constants c and C independent of ε and L such that one has the following properties for $j = \pm1, \ldots, \pm m$:

$$|\gamma_k - \gamma_k^{(j)}| \le \frac{C\varepsilon}{L^3}\Big|k - \frac{\zeta_j L}{\pi\varepsilon}\Big|^3\;, \qquad k \in K_1^{(j)}\;, \tag{7.7a}$$
$$|\gamma_k^{(j)}| \ge 1 + \frac{c}{L^2}\Big|k - \frac{\zeta_j L}{\pi\varepsilon}\Big|^2\;, \quad k \in K_1^{(j)}\;, \tag{7.7b}$$
$$|\gamma_k^{(j)}| \ge \frac{c}{\varepsilon^2}\;, \qquad k \in K_2 \setminus K_1^{(j)}\;, \tag{7.7c}$$
$$|\gamma_k^{(j)}| \ge ck^2/L^2\;, \qquad k \in K_3\;. \tag{7.7d}$$

In view of the series expansion of Lemma A.2, we also define

$$a_{n,k}^{(j)} = C\sqrt{\frac{1 - (-1)^n e^{-\gamma_k^{(j)}T}}{(\gamma_k^{(j)})^2 T^2 + \pi^2 n^2}}\;, \tag{7.8}$$

where the constant C depends only on T. We define $a_{n,k}$ in the same way with $\gamma_k^{(j)}$ replaced by γ_k. With these definitions at hand, we can use Lemma A.2 to write $\Phi^{(j)}$ as

$$\Phi^{(j)}(t, x) = \sqrt{\hat q(\zeta_j)} \sum_{k=-\infty}^{\infty} \sum_{n\in\mathbf{Z}} a_{n,k}^{(j)} \xi_{n,k}^{(j)} e_{n,k}^{(j)}(x, t)\;,$$

where we defined

$$e^{(j)}_{n,k}(x,t) = e_k(x)(e^{\frac{i\pi n}{T}t} - e^{-\gamma^{(j)}_k t})\;,$$

and where the $\{\xi^{(j)}_{n,k}\,:\; n \in \mathbf{Z}\}$ are independent complex-valued Gaussian random variables. Note that $e^{(-j)}_{-n,-k}(x,t) = \overline{e^{(j)}_{n,k}(x,t)}$, so that (7.6) implies the relation $\xi^{(-j)}_{-n,-k} = \overline{\xi^{(j)}_{n,k}}$. The process $W_{\mathcal{L}_\varepsilon}(t,x)$ can be expanded in a similar way as

$$W_{\mathcal{L}_\varepsilon}(t,x) = \sum_{k=-\infty}^{\infty} \sqrt{q^\varepsilon_k} \sum_{n\in\mathbf{Z}} a_{n,k}\xi_{n,k}e_{n,k}(x,t)\;, \tag{7.9}$$

with

$$e_{n,k}(x,t) = e_k(x)(e^{\frac{i\pi n}{T}t} - e^{-\gamma_k t})\;,$$

where $\{\xi_{n,k}\,:\; n \in \mathbf{Z}, k \in \mathbf{Z}\}$ are i.i.d standard complex-valued Gaussian random variables, with the exception that $\xi_{-n,-k} = \overline{\xi_{n,k}}$. Note that this implies that $\xi_{0,0}$ is real-valued. In order to be able to compare $W_{\mathcal{L}_\varepsilon}$ and Φ, we now specify how we choose the random variables $\xi_{n,k}$ to relate to the random variables $\xi^{(j)}_{n,k}$. For $j = \pm 1, \ldots, \pm m$ we define $\xi^{(j)}_{n,k} := \xi_{n,k}$ for all $k \in K^{(j)}_1$. Note that this is consistent with the relations $\xi^{(-j)}_{-n,-k} = \overline{\xi^{(j)}_{n,k}}$ and $\xi_{-n,-k} = \overline{\xi_{n,k}}$, and with the fact that $K^{(-j)}_1 = -K^{(j)}_1$. We will see later in the proof that the definition of $\xi^{(j)}_{n,k}$ for $k \notin K^{(j)}_1$ does not really matter, so we choose them to be independent of all the other variables, except for the relation $\xi^{(-j)}_{-n,-k} = \overline{\xi^{(j)}_{n,k}}$. The the proof of the proposition is split into several steps. First we bound the difference of $\frac{1}{2}\Pi^{(j)}_1\Phi^{(j)}$ and $\Pi^{(j)}_1 W_{\mathcal{L}_\varepsilon}$. Then we show that all remaining terms $(1-\Pi^{(j)}_1)\Phi^{(j)}$ and $(1-\Pi^{(0)}_1)W_{\mathcal{L}_\varepsilon}$ are small.

Step 1 We first prove that for $j = \pm 1, \ldots, \pm m$

$$\mathbb{E} \sup_{x\in[-L,L]} \sup_{t\in[0,T]} |\Pi^{(j)}_1\Phi^{(j)}(x,t) - \Pi^{(j)}_1 W_{\mathcal{L}_\varepsilon}(x,t)|^p \le C\varepsilon^{p/2-\kappa}\;. \tag{7.10}$$

We thus want to apply Lemma A.1 to

$$I(t,x) := \sum_{k\in K^{(j)}_1} \sum_{n\in\mathbf{Z}} \xi_{n,k}\big(\sqrt{\hat q(\zeta_j)}a^{(j)}_{n,k}e^{(j)}_{n,k}(x,t) - \sqrt{q^\varepsilon_k}a_{n,k}e_{n,k}(x,t)\big)$$

Define

$$f_{n,k}(x,t) = \sqrt{\hat q(\zeta_j)}a^{(j)}_{n,k}e^{(j)}_{n,k}(x,t) - \sqrt{q^\varepsilon_k}a_{n,k}e_{n,k}(x,t).$$

Note first that $\mathrm{Lip}(f_{n,k}) \le C(1+|k|+|n|+|\gamma_k|)$ and similarly for $\mathrm{Lip}(f^{(j)}_{n,k})$. Therefore, the uniform bounds on $\hat q$ and q^ε_k, together with the definition of $a_{n,\gamma}$ imply that there exists a constant C such that $\mathrm{Lip}(f_{n,k})$ is bounded by $C(|k|+1)$ for all $k \in K^j_1$ and $n \in \mathbf{N}$, where the constant only depends on T. Note that the Lipschitz constant is taken with respect to x and t. For $k \in K^{(j)}_1$ we have

$|k| \le C/\varepsilon$, and hence $\mathrm{Lip}(f_{n,k}) \le C\varepsilon^{-1}$. Now Lemma A.1 implies (7.10) if we can show that for every $\kappa > 0$ one has

$$\sum_{k\in K_1^{(j)}} \sum_{n\in\mathbf{Z}} \|f_{k,n}\|_\infty^{2-\kappa} \le C_\kappa \varepsilon^{1-\kappa} , \tag{7.11}$$

where the L^∞-norm is again taken with respect to t and x. To verify (7.11) we estimate $\|f_{k,n}\|_\infty$ by

$$\begin{aligned}\|f_{k,n}\|_\infty \le{}& |\sqrt{\hat q(\zeta_j)} - \sqrt{q_k^\varepsilon}||a_{n,k}|\|e_{n,k}\|_\infty + |\sqrt{\hat q(\zeta_j)}||a_{n,k}^{(j)}|\|e_{n,k}^{(j)} - e_{n,k}\|_\infty \\ &+ |\sqrt{\hat q(\zeta_j)}||a_{n,k}^{(j)} - a_{n,k}|\|e_{n,k}\|_\infty \\ =:{}& I_1(n,k) + I_2(n,k) + I_3(n,k) ,\end{aligned}$$

and we bound the three terms separately. First by assumption $\|\hat q\|_\infty \le C$. Furthermore, $a_{n,k} \le C/(1+|n|)$ and $\|e_{k,n}\|_\infty \le C$ for all $k \in K_1^{(j)}$ and $n \in \mathbf{N}$, and analogous for the terms involving j. Again by assumption $|\sqrt{\hat q(k_j)} - \sqrt{q_k^\varepsilon}| \le C\varepsilon$ for all $k \in K_1^{(j)}$, so that $I_1(n,k)$ is bounded by

$$|I_1(n,k)| \le \frac{C\varepsilon}{1+|n|} . \tag{7.12}$$

And hence, $\sum_{k,n} |I_1(n,k)|^{2-\kappa} \le C\varepsilon^{1-\kappa}$. For every $t > 0$ and every $\gamma' > \gamma > 0$

$$|e^{-\gamma t} - e^{-\gamma' t}| \le Ct|\gamma - \gamma'|e^{-\gamma t} .$$

Combining this with (7.7a) one has $\|e_{n,k}^{(j)} - e_{n,k}\|_\infty \le C\varepsilon|k - \frac{\zeta_j L}{\pi\varepsilon}|$ for $k \in K_1^{(j)}$. Using

$$\sum_{n=-\infty}^{\infty} (a_{n,k})^{2-\kappa} \le C \sum_{n=-\infty}^{\infty} (\gamma_k + |n|)^{\kappa-2} \le C/(\gamma_k(1+\gamma_k)),$$

we derive $\sum_{n=0}^{\infty} I_2(n,k)^{2-\kappa} \le C\varepsilon^{2-\kappa}$. Which gives the claim. Concerning I_3, a straightforward estimate using (7.7a) shows that

$$|I_3(n,k)| \le C|a_{n,k} - a_{n,k}^{(j)}| = C\varepsilon\frac{1 + \left|k - \frac{\zeta_j L}{\pi\varepsilon}\right|}{\gamma_k + |n|} .$$

Using $\sum_{n=-\infty}^{\infty}(\gamma_k + |n|)^{\kappa-2} \le C/(\gamma_k(1+\gamma_k))$ we derive $\sum_{n=-\infty}^{\infty} I_3(n,k)^{2-\kappa} \le \frac{C}{\gamma_k}\varepsilon^{2-\kappa}$, where we can use (7.7b). Combining all three estimates, bound (7.11) follows now easily.

Step 2 We now prove that

$$\mathbb{E} \sup_{x\in[-L,L]} \sup_{t\in[0,T]} |\Pi_3\Phi^{(j)}(x,t)|^p \le C\varepsilon^{p/2-\kappa} , \tag{7.13}$$

and

$$\mathbb{E} \sup_{x\in[-L,L]} \sup_{t\in[0,T]} |\Pi_3 W_{\mathcal{L}_\varepsilon}(x,t)|^p \le C\varepsilon^{p/2-\kappa} . \tag{7.14}$$

Both bounds are obtained in the same way, so we only show how to prove (7.14). Using the bound on q_k^ε, (7.8) and (7.7d) for $a_{n,k}$, and the definition of $e_{n,k}$, we readily obtain the bounds

$$\|q_k^\varepsilon a_{n,k} e_{n,k}\|_\infty \le \frac{C}{k^2+|n|} , \qquad \mathrm{Lip}(q_k^\varepsilon a_{n,k} e_{n,k}) \le Ck .$$

Now (7.14) follows immediately from Lemma A.1, noticing that

$$\sum_{n\in\mathbf{Z}}^{\infty} (k^2+|n|)^{-\delta} \le C|k|^{2-2\delta} , \quad \text{for } |k|\ge 1 \text{ and } \delta>1.$$

Furthermore, K_3 only contains elements k larger than $C\varepsilon^{-1}$.

Step 3 For $j = 0,\dots,m$ we denote by $\Pi_{21}^{(j)}$ the projector associated to the set $K_2 \setminus K_1^{(j)}$. We show that

$$\mathbb{E} \sup_{x\in[-L,L]} \sup_{t\in[0,T]} |\Pi_{21}^{(0)} W_{\mathcal{L}_\varepsilon}(x,t)|^p \le C\varepsilon^{p/2-\kappa} ,$$

and in a completely similar way we derive

$$\mathbb{E} \sup_{x\in[-L,L]} \sup_{t\in[0,T]} |\Pi_{21}^{(j)} \Phi^{(j)}(x,t)|^p \le C\varepsilon^{p/2-\kappa} .$$

By (7.8) and (7.7c) we get

$$\|q_k^\varepsilon a_{n,k} e_{n,k}\|_\infty \le \frac{C}{\varepsilon^{-2}+|n|} , \qquad \mathrm{Lip}(q_k^\varepsilon a_{n,k} e_{n,k}) \le C\varepsilon^{-1} .$$

The estimate follows then again from Lemma A.1, noticing that $K_2 - K_1$ contains less than $\mathcal{O}(\varepsilon^{-1})$ elements.

Summing up the estimates from all the previous steps concludes the proof. □

Appendix A Technical Estimates

A.1 Series expansion for stochastic convolutions

This section provides technical results on series expansion and their regularity of stochastic convolutions, which are necessary for the proofs.

Lemma A.1 *Let $\{\eta_k\}_{k\in I}$ be i.i.d. standard Gaussian random variables (real or complex) with $k \in I$ an arbitrary countable index set. Moreover let $\{f_k\}_{k\in I} \subset$*

$W^{1,\infty}(G,\mathbf{C})$ where the domain $G \subset \mathbf{R}^d$ has sufficiently smooth boundary (e.g. piecewise C^1). Suppose there is some $\delta \in (0,2)$ such that

$$S_1^2 = \sum_{k\in I} \|f_k\|_{L^\infty}^2 < \infty \quad \text{and} \quad S_2^2 = \sum_{k\in I} \|f_k\|_{L^\infty}^{2-\delta} \mathrm{Lip}(f_k)^\delta < \infty$$

Define $f(\zeta) = \sum_{k\in I} \eta_k f_k(\zeta)$. Then, with probability one, $f(\zeta)$ converges absolutely for any $\zeta \in G$ and, for any $p > 0$, there is a constant depending only on p, δ, and G such that

$$\mathbb{E}\|f\|_{C^0(G)}^p \le C(S_1^p + S_2^p) \,.$$

Proof. From the assumptions we immediately derive that $f(x)$ and $f(x) - f(y)$ are a centred Gaussian for any $x, y \in G$. Moreover, the corresponding series converge absolutely. Using that the η_k are i.i.d., we obtain

$$\begin{aligned} \mathbb{E}|f(x) - f(y)|^2 &= \sum_{k\in I} |f_k(x) - f_k(y)|^2 \\ &\le \sum_{k\in I} \min\{2\|f_k\|_{L^\infty}^2, \mathrm{Lip}(f_k)^2|x-y|^2\} \\ &\le 2\sum_{k\in I} \|f_k\|_{L^\infty}^{2-\delta}\mathrm{Lip}(f_k)^\delta|x-y|^\delta \\ &= 2S_2^2|x-y|^\delta \,, \end{aligned} \tag{A.1}$$

where we used that $\min\{a, bx^2\} \le a^{1-\delta/2}b^{\delta/2}|x|^\delta$ for any $a, b \ge 0$. Furthermore,

$$\mathbb{E}|f(x)|^2 \le \sum_{k\in I} \|f_k\|_{L^\infty}^2 = S_1^2 \,. \tag{A.2}$$

Consider $p > 1$ sufficiently large and $\alpha > 0$ sufficiently small. Using Sobolev embedding (cf. [Ada75, Theorem 7.57]) and the definition of the norm of the fractional Sobolev space in [Ada75, Theorem 7.48] we derive for $\alpha p > d$ that

$$\begin{aligned} \mathbb{E}\|f\|_{C^0(G)}^p &\le C\mathbb{E}\|f\|_{W^{\alpha,p}(G)}^p \\ &\le C\mathbb{E}\int_G\int_G \frac{|f(x)-f(y)|^p}{|x-y|^{d+\alpha p}}dxdy + C\mathbb{E}\int_G |f(x)|^p dx \\ &\le C\int_G\int_G \frac{(\mathbb{E}|f(x)-f(y)|^2)^{p/2}}{|x-y|^{d+\alpha p}}dxdy + C\int_G (\mathbb{E}|f(x)|^2)^{p/2}dx \,, \end{aligned}$$

where we used that $f(x)$ and $f(x) - f(y)$ are Gaussian. Note that the constants depend on p. Using (A.1) and (A.2), we immediately see that

$$\mathbb{E}\|f\|_{C^0(G)}^p \le CS_1^p + CS_2^p$$

provided $\alpha \in (0, \delta/2)$. Note finally that we needed $p > d/\alpha$ to have the Sobolev embedding available. The case of $p \le d/\alpha$ follows easily using Hölder inequality. □

Lemma A.2 *Let $\gamma \in$ **R** and let*

$$a(t) = \int_0^t e^{-\gamma(t-s)}\, dw(s)\;,$$

with w a standard complex Wiener process, i.e. $\mathbb{E}w(t)w(s) = 0$ and $\mathbb{E}w(t)\overline{w(s)} = \min\{t,s\}$. Then, for $t \in [0,T]$, $a(t)$ has the following representation:

$$a(t) = \sum_{n\in\mathbf{Z}} a_{n,\gamma}\xi_n\big(e^{\frac{\pi i n t}{T}} - e^{-\gamma t}\big)\;, \tag{A.3}$$

where the $a_{n,\gamma}$ are given by the Fourier-coefficients of $\frac{1}{2\gamma}e^{-\gamma|t-s|}$ on $[-T,T]$

$$a_{n,\gamma}^2 = C\frac{1-(-1)^n e^{-\gamma T}}{\gamma^2 T^2 + \pi^2 n^2}$$

with some constant C depending only on the time T. and the $\{\xi_n\}_{n\in\mathbf{Z}}$ are i.i.d. complex normal random variables, i.e. $\mathbb{E}\xi_n^2 = 0$ and $\mathbb{E}|\xi_n|^2 = 1$.

Proof. The stationary Ornstein–Uhlenbeck process

$$\tilde a(t) = \int_{-\infty}^t e^{-\gamma(t-s)}\, dw(s)$$

has the correlation function:

$$\mathbb{E}\overline{\tilde a(t)}\tilde a(s) = \frac{e^{-\gamma|t-s|}}{2\gamma}\;.$$

Expanding $e^{-\gamma|z|}$ in Fourier series on $[-T,T]$ we obtain

$$\tilde a(t) = \sum_{n\in\mathbf{Z}} a_{n,\gamma}\xi_n e^{i\pi n t/T}\;,$$

for i.i.d. normal complex-valued Gaussian random variables ξ_n. The claim now follows from the identity $a(t) = \tilde a(t) - e^{-\gamma t}\tilde a(0)$. □

A.2 A-priori estimate for the amplitude equation

This section summarises and proves technical a-priori estimates for an equation of the type (GL). Most of them are obtained by standard methods and the proofs will be omitted. The main non-trivial result is Theorem A.7 about the concentration in Fourier space. We consider the equation

$$\partial_t A = \alpha\partial_x^2 A + i\beta\partial_x A + \gamma A - c|A|^2 A + \sigma\eta \tag{A.4}$$

with periodic boundary conditions on $[-L,L]$, where α and c are positive and $\sigma,\gamma,\beta \in$ **R** and η denotes space–time white noise.

Equation (GL) is of the form (A.4) with $\alpha = 4$, $\beta = -8\delta_\varepsilon$, $\gamma = \nu - 4\delta_\varepsilon$ and $c = 3$ with $|\delta_\varepsilon| \leq \frac{\pi}{2L}$. Obviously, the constants β and γ are ε-dependent, but uniformly bounded in $\varepsilon > 0$, which is a straightforward modification of the result presented.

Further, we denote by $\mathcal{W}$ the complex cylindrical Wiener process such that $\partial_t \mathcal{W} = \eta$. Define the stochastic convolution

$$\varphi = \sigma \mathcal{W}_{\alpha\partial_x^2 - 1} \qquad \text{and} \qquad B = A - \varphi. \tag{A.5}$$

Then

$$\partial_t B = \alpha\partial_x^2 B + i\beta\partial_x(B+\varphi) + \gamma B + (\gamma+1)\varphi - c|B+\varphi|^2(B+\varphi). \tag{A.6}$$

Of course this equation is only formal, as φ is not differentiable. But in what follows, we can always use smooth approximations of φ to justify the arguments. The mild formulation of (A.6) is

$$\begin{aligned} B(t) = e^{\alpha\partial_x^2 t}A(0) + i\beta \int_0^t \partial_x e^{\alpha\partial_x^2(t-s)}(B+\varphi)(s)ds \qquad (A.7) \\ + \int_0^t e^{\alpha\partial_x^2(t-s)}\Big(\gamma B(s) + (\gamma+1)\varphi(s) - c|B+\varphi|^2(B+\varphi)(s)\Big)ds\ . \end{aligned}$$

We will use the following Lemma, which fails to be true in higher dimensions for complex space-time white noise η.

Lemma A.3 *For any choice of $q \geq 1$ and $T_0 > 0$ there are constants such that*

$$\sup_{t\in[0,T_0]} \mathbb{E}\|\varphi(t)\|^q_{\mathcal{C}^0_a} \leq C \quad \text{and} \quad \mathbb{E} \sup_{t\in[0,T_0]} \|\varphi(t)\|^q_{\mathcal{C}^0_a} \leq C.$$

The results of the previous lemma are obviously also true if we replace the C^0-norm by an L^p-norm. The constant then depends also on p. The proof of this lemma is standard see *e.g.* [BH04] or [BMPS01, Theorem 5.1.]. Now we easily prove the following result via standard energy-type estimates for $A - \varphi$.

Proposition A.4 *For any choice of $p \geq 1$, $q \geq 1$, and $T_0 > 0$ there are constants such that*

$$\sup_{t\geq T_0} \mathbb{E}\|A(t)\|^q_{L^p_a} \leq C,$$

with constant independent of $A(0)$. Moreover, for any choice of $c_0 > 0$, $p \geq 1$, $q \geq 1$, and $T_0 > 0$ there are constants such that if $\|A(0)\|^q_{L^p_a} \leq c_0$ then

$$\sup_{t\in[0,T_0]} \mathbb{E}\|A(t)\|^q_{L^p_a} \leq C \quad \text{and} \quad \mathbb{E} \sup_{t\in[0,T_0]} \|A(t)\|^q_{L^p_a} \leq C.$$

Now we can easily verify the following result using the mild formulation of solutions.

Proposition A.5 *For any choice of $c_0 > 0$, $q \geq 1$, and $T_0 > 0$ there are constants such that if $\mathbb{E}\|A(0)\|_{\mathcal{C}_a^0}^{3q} \leq c_0$ then*

$$\mathbb{E} \sup_{t\in[0,T_0]} \|A(t)\|_{\mathcal{C}_a^0}^q \leq C.$$

Note that it is sufficient for Proposition A.5 to assume that $A(0)$ is admissible.

Remark A.6 We need the condition on the $3q$th moment of the initial conditions to ensure that $\mathbb{E}\sup_{t\in[0,T_0]} \|B|B|^2(t)\|_{L_a^p}^q \leq C$.

In the following we establish that a solution A of (A.4) with admissible initial conditions, in the sense of Definition 3.4, stays concentrated in Fourier space in the $\mathcal{C}^0$-topology for all times.

Theorem A.7 *Let $A(t)$ be the solution of (A.4) and assume that the initial conditions are admissible. Then for every $p \geq 1$ and $T_0 > 0$ there exist positive constants κ, C_0 with $\kappa \leq 1$ such that*

$$\mathbb{E} \sup_{t\in[0,T_0]} \|\Pi_{\delta/\varepsilon}^c A(t)\|_{\mathcal{C}_a^0}^p \leq C\varepsilon^{p/2-\kappa} ,$$

where $\Pi_{\delta/\varepsilon}^c$ was defined in (2.7).

Proof. We start by establishing the fact that admissible initial conditions are concentrated in Fourier space. According to Definition 3.4 the initial conditions admit the decomposition $A(0) = W_0 + A_1$. Consider first the Gaussian part W_0. We can use the series expansion of Remark 3.6 together with Lemma A.1 to verify

$$\mathbb{E}\|\Pi_{\delta/\varepsilon}^c W_0\|_{\mathcal{C}_a^0}^p \leq C_p\varepsilon^{p/2-\kappa}.$$

Let now $\{A_k^1\}_{k\in\mathbb{Z}}$ denote the Fourier coefficients of A_1. We use the fact that A_1 is bounded in $\mathcal{H}_a^1$ to deduce

$$\begin{aligned}\|\Pi_{\delta/\varepsilon} A_1\|_{\mathcal{C}_a^0}^2 &\leq \Big(\sum_{|k|\geq\frac{\delta}{\varepsilon}} |A_k^1|\Big)^2 \leq \sum_{|k|\geq\frac{\delta}{\varepsilon}} |k|^{-2} \sum_{k\in\mathbb{Z}} |k|^2|A_k^1|^2 \\ &\leq C\varepsilon^{1-\kappa}\|A_1\|_{a,1}^2 .\end{aligned}$$

From the above estimates we deduce that

$$\mathbb{E}\|\Pi_{\delta/\varepsilon}^c A(0)\|_{\mathcal{C}_a^0}^p \leq C\varepsilon^{p/2-\kappa} .$$

Let us consider (A.7). First using the boundedness of the semigroup

$$\mathbb{E}\|\Pi_{\delta/\varepsilon}^c e^{\alpha t\partial_x^2} A(0)\|_{\mathcal{C}_a^0}^p \leq C\mathbb{E}\|\Pi_{\delta/\varepsilon}^c A(0)\|_{\mathcal{C}_a^0}^p \leq C\varepsilon^{p/2-\kappa} .$$

Using the factorisation method (see *e.g.* [BMPS01, Theorem 5.1.]) we easily get for the stochastic convolution φ defined in (A.5) the bound

$$\mathbb{E}\Big\|\sup_{t\in[0,T_0]}\Pi^c_{\delta/\varepsilon}\varphi(t)\Big\|_a^p \le C\Big(\sum_{|k|\ge\delta/\varepsilon}|k|^{-2+2\kappa}\Big)^{p/2} \le C\varepsilon^{p/2-\kappa}. \tag{A.8}$$

To proceed, we use the stability of the semigroup and the embedding of $\mathcal{H}^\zeta$ into $\mathcal{C}^0_a$ for $\zeta\in(\frac12,1)$. Using this, it is elementary to show that

$$\|\Pi^c_{\delta/\varepsilon}e^{t\alpha\partial_x^2}h\|_{\mathcal{C}^0_a} \le Ce^{-ct\varepsilon^{-2}}t^{-\zeta/2}\|h\|_a\;,$$

for every $h\in\mathcal{H}_a$. Hence

$$\begin{aligned}\Big\|\Pi^c_{\delta/\varepsilon}\int_0^t e^{(t-s)\partial_x^2}h(s)\,ds\Big\|_{\mathcal{C}^0_a} &\le C\int_0^t e^{-Cs\varepsilon^{-2}}s^{-\alpha/2}ds\sup_{s\in[0,T]}\|h(s)\|_a\\ &\le C\varepsilon^{2-\zeta}\sup_{s\in[0,T]}\|h(s)\|_a\;.\end{aligned}$$

Moreover, for $h=\sum h_ke_k$ by a crude estimate

$$\begin{aligned}\Big\|\Pi^c_{\delta/\varepsilon}\partial_x\int_0^t e^{(t-s)\alpha\partial_x^2}h(s)\,ds\Big\|_{\mathcal{C}^0_a} &\le \sum_{|k|\ge\delta/\varepsilon}\int_0^t|k|e^{-c(t-s)k^2}|h_k(s)|ds\\ &\le C\int_0^t e^{-Cs\varepsilon^{-2}}s^{-(1+\zeta)/2}ds\sup_{s\in[0,t]}\|h(s)\|_a\\ &\le C\varepsilon^{1-\zeta}\sup_{s\in[0,t]}\|h(s)\|_a\;.\end{aligned}$$

Using (A.7), Proposition A.5, and (A.8) and choosing $\zeta>\frac12$ sufficiently small (*e.g.* $\zeta=\frac12+\frac{\kappa}{p}$), it is now straightforward to verify the assertion first for B and hence for A. □

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